Reproducing a Type of Aspero-Mota Iteration

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Abstract

We reproduce a type of Aspero-Mota iteration of finitely proper posets in two stages. In the first stage, we force a type of stationary set $\mathcal{E}$ to the set of countable subsets of κ (i.e., the length of the iteration) by proper forcing. In the second stage, we iteratively force with $\mathcal{E}$-finitely proper posets that satisfy a size restriction. Subsequently, the iteration thus constructed is considered $\mathcal{E}$-proper. In this way, $\mathcal{E}$ guides the second stage of our construction.

Since relevant posets possess appropriate chain conditions, we deal with them as predicates in a first-order structure $\mathcal{A}$ whose universe is a common transitive set H_κ, i.e., the set of sets that are hereditarily of sizes less than κ. Since elementary substructures of $\mathcal{A}$ do not contain our posets as points but as predicates in the substructures, careful treatments of forcing that regards posets as predicates are required.

In this note, we prepare basics in this line of our treatment of forcing. Subsequently, we prepare the stationary set $\mathcal{E}$ in the first stage of our construction. The remainder of our construction will be continued in a sequel to this note.

Introduction

Aspero and Mota introduced a new type of iterated forcing in [AM]. They formulated a class of posets called finitely proper. This class includes the ccc posets. They iteratively force with posets that are finitely proper whose underlining sets are of sizes at most the least uncountable cardinal ω_1. While performing this process, they used side conditions. A side condition to a condition p is a finite relation R from the elementary substructures N of a prefixed first-order structure $\mathcal{A}$, whose universe is the transitive set H_κ, to the stages ξ of the iteration. A combination of a type of crucial elementary substructure N and a stage $\xi \in N$, which is known as a marker of N, in R indicates that p is (P_ξ, N)-generic

Aspero and Mota treat crucial Ns as projections of N^*s that are derived from simple-looking clubs with respect to various considerably large transitive sets. Since relevant posets possess the ω_2-cc, we treat our posets as predicates to expand $\mathcal{A}$. Hence, we are free from formulating rather abstract projections. Furthermore, there is a possibility of having more control over the relevant countable elementary substructures. However, we must cautiously treat iterated forcing to regard posets as predicates.

§ 1. Basics

Throughout this paper, we have a fixed regular cardinal $\kappa \geq \omega_2$. Let P be a poset such that $P \subseteq H_\kappa$ and P has the κ-cc.

1.1 Lemma. (1) For any P-name τ, there exists a P-name $\pi \in H_\kappa$ such that

$$1 \Vdash_P \text{“if } \tau \in H_\kappa^{V[\dot{G}]}, \text{ then } \pi = \tau\text{”}.$$

(2) If $\tau \in H_\kappa$ is a P-name, then $1 \Vdash_P$“$\tau \in H_\kappa^{V[\dot{G}]}$”.

(3) $1 \Vdash_P$“$H_\kappa^{V[\dot{G}]} = \{\tau_{\dot{G}} \mid \tau \in H_\kappa^V$ is a P-name$\}$”.

Proof. (1) : By induction on $\tau_i \in$ dom(τ). Let us enumerate members of τ such that

$$\tau = \{\cdots, (\tau_i, p_i), \cdots\}.$$

Let us take a P-name $\pi_i \in H_\kappa$ corresponding to τ_i for each i.
Since P has the κ-cc, we have a set of ordinals I such that $|I| < \kappa$ and

$$1 \Vdash_P \text{“if } \tau \in H_\kappa^{V[\dot{G}]}, \text{ then } \tau = \{(\pi_i)_{\dot{G}} \mid i \in I,\ p_i \in \dot{G}\}\text{”}.$$

Let $\pi = \{(\pi_i, p_i) \mid i \in I\}$. Then $\pi \in H_\kappa$ is a P-name such that

$$1 \Vdash_P \text{“}\pi = \{(\pi_i)_{\dot{G}} \mid i \in I,\ p_i \in \dot{G}\}\text{”}.$$

Hence $1 \Vdash_P$“if $\tau \in H_\kappa^{V[\dot{G}]}$, then $\pi = \tau$”.

(2) : Let us enumerate $\tau \in H_\kappa \cap V^P$ such that $\tau = \{(\tau_i, p_i) \mid i \in I\}$ with $|I| < \kappa$. By induction we have $1 \Vdash_P$“$\tau_i \in H_\kappa^{V[\dot{G}]}$”.
Then $1 \Vdash_P$“$\tau = \{(\tau_i)_{\dot{G}} \mid i \in I,\ p_i \in \dot{G}\} \subset H_\kappa^{V[\dot{G}]}$”.
Since κ remains regular, we have $1 \Vdash_P$“$\tau \in H_\kappa^{V[\dot{G}]}$”

(3) : By (1) and (2).

□

1.2 Lemma. For any formula $\varphi(x_1, \cdots, x_n)$, there exists a formula $\varphi^*(y, x_1, \cdots, x_n)$ such that for any sequence of P-names $\tau_1, \cdots, \tau_n$ and any $p \in P$, we have

$$p \Vdash_P \text{“}(H_\kappa^{V[\dot{G}]}, \in, H_\kappa^V, \dot{G}, P, \leq, 1, R_{=}^{P}) \models \text{“}\varphi(\tau_1, \cdots, \tau_n)\text{””}$$

iff

$$(H_\kappa, \in, P, \leq, 1, R_{=}^{P}) \models \text{“}\varphi^*(p, \tau_1, \cdots, \tau_n)\text{”}.$$

Here, $R^P_= = \{(q, \tau, \pi) \mid q \in P, \tau, \pi \in H_\kappa \cap V^P$ such that $q \Vdash_P$ "$\tau = \pi$"$\}$ is a ternary relation on H_κ that is not definable in the structure $(H_\kappa, \in, P, \leq, 1)$ in general.

Proof. By induction on φ. We discuss typical cases and employ abbreviations in the rest.

<u>*atomic*</u>: $p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\tau \in \pi$"" iff $\forall d \leq p \exists r \leq d \exists (\pi_i, p_i) \in \pi$ such that $r \leq p_i$ and $r \Vdash_P$ "$\tau = \pi_i$". $p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "τ is in H_κ^V"" iff $\forall d \leq p \exists r \leq d \exists e, z \in H_\kappa$ such that $z = \check{e}$ and $r \Vdash_P$ "$\tau = z$", where $z = \check{e}$ iff $\exists f : \mathrm{TC(e)} \longrightarrow H_\kappa$ such that $\forall x \in \mathrm{TC}(e) f(x) = \{(f(y), 1) \mid y \in x\}$ and $z = \{(f(x), 1) \mid x \in e\}$.

$p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "τ is in $\dot{G}$"" iff $\forall d \leq p \exists r \leq d \exists g\, r \leq g$ such that $r \Vdash_P$"$\tau = \check{g}$".

$p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "τ is in P"" iff $\forall d \leq p \exists r \leq d \exists g \in P$ such that $r \Vdash_P$"$\tau = \check{g}$".

$p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\tau \leq \pi$"" iff $\forall d \leq p \exists r \leq d \exists g_1, g_2 \in P$ such that $g_1 \leq g_2$ and $r \Vdash_P$"$\tau = \check{g}_1$ and $\pi = \check{g}_2$".

$p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$R^P_=(\tau, \pi_1, \pi_2)$"" iff $\forall d \leq p \exists r \leq d \exists g, g_1, g_2 \in P$ such that $R^P_=(g, g_1, g_2)$ and $r \Vdash_P$"$\tau = \check{g}$, $\pi_1 = \check{g}_1$, and $\pi_2 = \check{g}_2$".

We discuss typical atomic cases where the constant 1 gets involved. Let φ be any atomic formula such that $p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\varphi(\tau, \pi)$"" iff $H_\kappa \models$ "$\varphi^*(p, \tau, \pi)$" Then $p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\varphi(\check{1}, \pi)$"" iff $H_\kappa \models$ "$\varphi^*(p, \check{1}, \pi)$" iff $\exists \tau \in H_\kappa$ such that $\tau = \check{1}$ and $H_\kappa \models$ "$\varphi^*(p, r, \pi)$"

<u>*logic*</u>: We discuss typical two cases.

$p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\varphi_1(\tau) \wedge \varphi_2(\tau)$"" iff $p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\varphi_1(\tau)$"" and $p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\varphi_2(\tau)$"" iff $H_\kappa \models$ "$\varphi_1^*(p, \tau)$" and $H_\kappa \models$ "$\varphi_1^*(p, \tau)$" iff $H_\kappa \models$ "$\varphi_1^*(p, \tau) \wedge \varphi_2^*(p, \tau)$".

$p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\neg\varphi(\tau)$"" iff $\forall d \leq p (d \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\varphi(\tau)$""gets negated) iff $\forall d \leq p (H_\kappa \models$ "$\varphi^*(d, \tau)$"gets negated) iff $\forall d \leq p\ H_\kappa \models$ "$\neg\varphi^*(d, \tau)$".

<u>*exists*</u>: $p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\exists y \varphi(y, \tau)$"" iff $\exists \dot{y} \in H_\kappa \cap V^P$ such that $p \Vdash_P$"$H_\kappa^{V[\dot{G}]} \models$ "$\varphi(\dot{y}, \tau)$"" iff $\exists \dot{y} \in H_\kappa \cap V^P\ H_\kappa \models$ "$\varphi^*(p, \dot{y}, \tau)$", where

$\dot{y} \in H_\kappa \cap V^P$ iff $\exists f : \mathrm{TC}(\dot{y}) \longrightarrow 2$ such that $\forall x \in TC(\dot{y})$, $f(x) = 1$ iff x is a binary relation such that $f[\mathrm{dom}(x)] = \{1\}$ and $\mathrm{range}(x) \subseteq P$, and $\dot{y}$ is a binary relation such that $f[\mathrm{dom}(\dot{y})] = \{1\}$ and $\mathrm{range}(\dot{y}) \subseteq P$.

$\square$

1.3 Lemma. Let N be an elementary substructure of the structure $(H_\kappa, \in, P, \leq, 1, R^P_=)$, this be denoted by

$$N \prec (H_\kappa, \in, P, \leq, 1, R^P_=).$$

Let us denote

$$1 \Vdash_P \text{"}N[\dot{G}] = \{\tau_{\dot{G}} \mid \tau \in N \cap V^P\}\text{"}.$$

Then we have

$$1 \Vdash_P \text{"}N[\dot{G}] \prec (H_\kappa^{V[\dot{G}]}, \in, H_\kappa^V, \dot{G}, P, \leq, 1, R^P_=)\text{"}.$$

Proof. Let $\tau \in N \cap V^P$. Suppose that

$$p \Vdash_P \text{“}H_\kappa^{V[\dot{G}]} \models \text{“}\exists y \varphi(y, \tau)\text{””}.$$

It suffices to get $\dot{y} \in N \cap V^P$ such that

$$p \Vdash_P \text{“}H_\kappa^{V[\dot{G}]} \models \text{“}\varphi(\dot{y}, \tau)\text{””}.$$

However

$$1 \Vdash_P \text{“}H_\kappa^{V[\dot{G}]} \models \text{“}\exists y \forall z (\varphi(z, \tau) \Longrightarrow \varphi(y, \tau))\text{”}.$$

Hence

$$\exists \dot{y} \in H_\kappa \cap V^P \forall \dot{z} \in H_\kappa \cap V^P 1 \Vdash_P \text{“}H_\kappa^{V[\dot{G}]} \models \text{“}\varphi(\dot{z}, \tau) \Longrightarrow \varphi(\dot{y}, \tau)\text{”}.$$

Let Φ^* be a formula such that for any P-names $\dot{y}$, $\dot{z}$, π in H_κ, and any $q \in P$, we have

$$q \Vdash_P \text{“}H_\kappa^{V[\dot{G}]} \models \text{“}\varphi(\dot{z}, \pi) \Longrightarrow \varphi(\dot{y}, \pi)\text{”}.$$

iff

$$H_\kappa \models \text{“}\Phi^*(q, \dot{y}, \dot{z}, \pi)\text{”}.$$

Then

$$\exists \dot{y} \in H_\kappa \cap V^P \forall \dot{z} \in H_\kappa \cap V^P \ H_\kappa \models \text{“}\Phi^*(1, \dot{y}, \dot{z}, \tau)\text{”}.$$

Rewrite this as

$$H_\kappa \models \text{“}\exists \dot{y}\ P\text{-name}\ \ \forall \dot{z}\ P\text{-name}\ \Phi^*(1, \dot{y}, \dot{z}, \tau)\text{”}.$$

Since $N \prec H_\kappa$, there exists $\dot{y} \in N \cap V^P$ such that

$$H_\kappa \models \text{“}\forall \dot{z}\ P\text{-name}\ \Phi^*(1, \dot{y}, \dot{z}, \tau)\text{”}.$$

Hence

$$\forall \dot{z} \in H_\kappa \cap V^P 1 \Vdash_P \text{“}H_\kappa^{V[\dot{G}]} \models \text{“}\varphi(\dot{z}, \tau) \Longrightarrow \varphi(\dot{y}, \tau)\text{””}.$$

Rewrite this as

$$1 \Vdash_P \text{“}H_\kappa^{V[\dot{G}]} \models \text{“}\forall z (\varphi(z, \tau) \Longrightarrow \varphi(\dot{y}, \tau))\text{””}.$$

Hence

$$p \Vdash_P \text{“}H_\kappa^{V[\dot{G}]} \models \text{“}\varphi(\dot{y}, \tau)\text{””}.$$

□

1.4 Lemma. Let $N \prec (H_\kappa, \in, P, \leq 1, R^P_=)$ be countable. The following are equivalent.

(1) For all *predense* subsets $A \in N$ of P, $A \cap N$ are *predense* below p.

(2) $p \Vdash_P \text{“}N[\dot{G}] \cap H_\kappa^V = N\text{”}$.

(3) $p \Vdash_P \text{“}N[\dot{G}] \cap \kappa = N \cap \kappa\text{”}$.

Proof. (1) $\Longrightarrow$ (2): Let $\tau \in N \cap V^P$. Let A be a maximal antichain such that

$$A \subseteq \{a \in P \mid a \Vdash_P \text{“}\tau \notin H_\kappa^V\text{”} \parallel \exists e \in H_\kappa^V a \Vdash_P \text{“}\tau = \check{e}\text{”}\}.$$

Since P has the κ-cc, we may assume that $A \in N$. Hence $A \cap N$ is predense below p. Let G be P-generic over the ground model V with $p \in G$. Then in $V[G]$, we have $a \in A \cap N \cap G$. Since we assume that $\tau_G \in N[G] \cap H_\kappa^V$, it must be the case that there exists $e \in H_\kappa^V$ such that $a \Vdash_P$ “$\tau = \check{e}$”. Since a, $\tau \in N \prec H_\kappa$, we may take $e \in N$. Hence $\tau_G = e \in N$. This establishes that $N[G] \cap H_\kappa^V \subseteq N$. Conversely, let $x \in N$. Then $\check{x} \in N \cap V^P$ and so $x \in N[G] \cap H_\kappa^V$.

(2) $\Longrightarrow$ (3): Since $\kappa \subset H_\kappa^V$, this is trivial.

(3) $\Longrightarrow$ (1): Let $A \in N$ be a predense subset of P. Let us enumerate $f: |A| \longrightarrow A$. We may assume $f \in N \prec H_\kappa$. Let G be P-generic over V with $p \in G$. Then in $V[G]$, there exists $i \in |A|$ such that $f(i) \in G$. Since $N[G] \prec H_\kappa^{V[G]}$, we may assume that $i \in N[G] \cap \kappa = N \cap \kappa$. Hence $f(i) \in A \cap N \cap G \neq \emptyset$. This establishes that $A \cap N$ is predense below p.

$\square$

If a poset P is in H_κ, we have a usual treatment by considering $P \in N \prec (H_\kappa, \in)$. But two appraoches are equivalent.

1.5 Lemma. Let $(P, \leq, 1) \in H_\kappa$ be a poset. Then the following are equivalent.

(1) $(P, \leq, 1) \in N \prec (H_\kappa, \in)$.

(2) $N \prec (H_\kappa, \in, P, \leq, 1, R_{=}^P)$.

Proof. We comment on a harder direction (1) $\Longrightarrow$ (2):

Claim. For any formula $\varphi(v_1, \cdots, v_n)$, there exists a formula $\varphi^-(u, v, w, v_1, \cdots, v_n)$ such that for any sequence $x_1, \cdots, x_n \in H_\kappa$, we have

$$(H_\kappa, \in, P, \leq, 1, R_{=}^P) \models \text{“}\varphi(x_1, \cdots, x_n)\text{”}$$

iff

$$(H_\kappa, \in) \models \text{“}\varphi^-(P, \leq, 1, x_1, \cdots, x_n)\text{”}.$$

To see this, we observe that the ternary relation $R_{=}^P$ is definable in $(H_\kappa, \in)$ with the parameter $(P, \leq, 1)$ by considering its characteristic function $f \in H_\kappa$. We follow [K].

$p \Vdash_P$ “$\tau_1 = \tau_2$” iff $\exists f: P \times (\mathrm{TC}(\tau_1) \cap V^P) \times (\mathrm{TC}(\tau_2) \cap V^P) \longrightarrow 2$ such that (1), (2), and (3);

(1) $f(p', \pi_1, \pi_2) = 1$ iff (a) and (b);

(a) $\forall (\pi_1', s_1') \in \pi_1 \forall d \leq p' \exists q \leq d$ such that $(q \not\leq s_1' \parallel \exists (\pi_2', s_2') \in \pi_2 q \leq s_2' f(q, \pi_1', \pi_2') = 1)$.

(b) $\forall (\pi_2', s_2') \in \pi_2 \forall d \leq p' \exists q \leq d$ such that $(q \not\leq s_2' \parallel \exists (\pi_1', s_1') \in \pi_1 q \leq s_1' f(q, \pi_1', \pi_2') = 1)$.

(2) $\forall (\pi_1, s_1) \in \tau_1 \forall d \leq p \exists q \leq d$ such that $(q \not\leq s_1 \parallel \exists (\pi_2, s_2) \in \tau_2 q \leq s_2 f(q, \pi_1, \pi_2) = 1)$.

(3) $\forall (\pi_2, s_2) \in \tau_2 \forall d \leq p \exists q \leq d$ such that $(q \not\leq s_2 \parallel \exists (\pi_1, s_1) \in \tau_1 q \leq s_1 f(q, \pi_1, \pi_2) = 1)$.

$\square$

We consider basics related to two step iterations $P*\dot{Q}$. We deal with a relevant case where $\dot{Q}$ satisfies a size restriction. In particular, it is a point in H^P_κ.

1.6 Lemma. Let $\Vdash_P$“$\dot{Q}=(\omega_1^V, \dot{\leq}, \dot{1})$ be a poset”. We form a two step iteration $P*\dot{Q}$ such that

$$\{(p, \dot{1}) \mid p\in P\} \cup \{(p, \check{i}) \mid p \Vdash_P \text{“}\check{i}\in\dot{Q}\text{”}\} \subset P*\dot{Q} \subset H_\kappa.$$

Let $X \prec (H_\kappa, \in, P*\dot{Q}, \leq_{P*\dot{Q}}, (1_P, \dot{1}), R^{P*\dot{Q}}_{=})$. Then we have

(1) $X \prec (H_\kappa, \in, P, \leq_P, 1_P, R^P_{=})$.

(2) $1_P \Vdash_P$“$\dot{Q}\in X[\dot{G}] \prec (H_\kappa^{V[\dot{G}]}, \in, H^V_\kappa, \dot{G}, P, \leq_P, 1_P)$”.

Proof. (1) : We first

Claim. For may formula $\varphi(v_1, \cdots, v_n)$, there exists a formula $\varphi^+(v_1, \cdots, v_n)$ such that for any sequence $x_1, \cdots, x_n \in H_\kappa$, we have

$$(H_\kappa, \in, P, \leq_P, 1_P, R^P_{=}) \models \text{“}\varphi(x_1, \cdots, x_n)\text{”}.$$

iff

$$(H_\kappa, \in, P*\dot{Q}, \leq_{P*\dot{Q}}, (1_P, \dot{1}), R^{P*\dot{Q}}_{=}) \models \text{“}\varphi^+(x_1, \cdots, x_n)\text{”}.$$

To see this, we consider a translation of P-names τ to $P*\dot{Q}$-names τ^*. such that $\Vdash_{P*\dot{Q}}$“$\check{\tau}_{\dot{G}\lceil\check{P}}=\tau^*$”. The translation is definable in $(H_\kappa, \in, P*\dot{Q}, \leq_{P*\dot{Q}}, (1_P, \dot{1}))$. Namely, $y=\tau^*$ iff $\exists t : \mathrm{TC}(\tau)\cap V^P \longrightarrow V^{P*\dot{Q}}\cap H_\kappa$ such that

$$t(\tau')=\{(t(\pi'), (s', \dot{1})) \mid (\pi', s')\in\tau'\}.$$
$$y=\{(t(\pi), (s, \dot{1})) \mid (\pi, s)\in\tau\}.$$

(2) : By (1), we have $1_P \Vdash_P$“$X[\dot{G}] \prec (H_\kappa^{V[\dot{G}]}, \in, H^V_\kappa, \dot{G}, P, \leq_P, 1_P)$”. To see $\dot{\leq}\in X[\dot{G}]$, we must have its P-name in X. But we may pick a sequence $\langle A_{ij} \mid i, j\in\omega_1\rangle\in X$ such that A_{ij} is a maximal antichain in $\{a\in P \mid (a, \check{i}) \leq_{P*\dot{Q}} (a, \check{j})\}$. Then we see that $\{(a, (i, j)) \mid a\in A_{ij}, i, j<\omega_1\}\in X\cap V^P$ is a P-name of $\dot{\leq}$.

□

§2. Mother $\mathcal{E}$

We discuss the first stage of [AM] where what they call a symmetric system gets forced. We focus on the family of countable subsets of κ induced from the system.

2.1 Definition. We say a stationary subset $\mathcal{E}$ of $[\kappa]^\omega$ is a *mother*, if

(1) If $X\in\mathcal{E}$ and $Y\in\mathcal{E}\lceil X$, then $Y\cap\omega_1 < X\cap\omega_1 < \omega_1$, where $\mathcal{E}\lceil X=\{Y\in\mathcal{E} \mid Y\neq X, Y\subset X\}$.

(2) If $X_1, X_2\in\mathcal{E}$ with $X_1\cap\omega_1=X_2\cap\omega_1$, then there exists an isomorphism $\phi : (X_1, <) \longrightarrow (X_2, <)$ such that $\phi\lceil(X_1\cap X_2)$ is the identity and $\mathcal{E}\lceil X_2=\phi[\mathcal{E}\lceil X_1]=\{\phi[Y_1] \mid Y_1\in\mathcal{E}\lceil X_1\}$.

(3) If $X_1, Y_2\in\mathcal{E}$ with $X_1\cap\omega_1 < Y_2\cap\omega_1$, then there exists $Y_1\in\mathcal{E}$ such that $Y_1\cap\omega_1=Y_2\cap\omega_1$

and $X_1 \in \mathcal{E} \lceil Y_1$.

(4) If X, $Y \in \mathcal{E}$ with $Y \cap \omega_1 < X \cap \omega_1$ and $\alpha \in X$ with $\mathrm{cf}(\alpha) \geq \omega_1$, then there exists $\rho \in X \cap \alpha$ such that $X \cap Y \cap \alpha \subset \rho$.

2.2 Lemma. (CH) There exists a poset P such that

(1) $P \subset H_\kappa$ is proper and has the ω_2-cc.

(2) In the generic extensions V^P, there exists a mother.

The relevant poset is the very first stage of [AM].

2.3 Definition. $p \in P_{AM} = P$, if

- (ob) p is a finite set such that for all $N \in p$, $N \prec (H_\kappa, \in)$ is countable.
- (iso) If N_1, $N_2 \in p$ with $N_1 \cap \omega_1 = N_2 \cap \omega_1$, then there exists an isomorphism

$$\phi : (N_1, \in, p \cap N_1) \longrightarrow (N_1, \in, p \cap N_2)$$

such that $\phi \lceil (N_1 \cap N_2)$ is the identity.

- (up) If N, $M \in p$ with $M \cap \omega_1 < N \cap \omega_1$, then there exists $Z \in p$ such that $M \in Z$ and $Z \cap \omega_1 = N \cap \omega_1$.

For p, $q \in P$, define $q \leq p$, if $q \supseteq p$.

The important facts in taking copies of countable sets are the following.

2.4 Lemma. Let N_1, $N_2 \prec (H_\kappa, \in)$, say both countable, $\phi : (N_1, \in) \longrightarrow (N_2, \in)$ be an isomorphism, and a countable set $X \in N_1$ so that $\phi(X) = \phi[X]$. Then

(1) (uniqueness) If $c_{N_1} : (N_1, \in) \longrightarrow (\overline{N_1}, \in)$, $c_{N_2} : (N_2, \in) \longrightarrow (\overline{N_2}, \in)$ are the unique transitive collapses, then $c_{N_2} \circ \phi = c_{N_1}$ and so $\phi = c_{N_2}^{-1} \circ c_{N_1}$ is unique.

(2) (copying elementarity) If $X \prec (H_\kappa, \in)$, then $\phi(X) \prec (H_\kappa, \in)$.

(3) (copying iso) If $f : (X, \in) \longrightarrow (Y, \in)$, say both X and Y are countable, is an isomorphism such that f, X, $Y \in N_1$, then $\phi(f) : (\phi(X), \in) \longrightarrow (\phi(Y), \in)$ is an isomorphism.

Proof. (1) : The axiom of extensionality gets satisfied by both N_1 and N_2. Hence we have the uniqueness of transitive collapses.

(2) : Let $x \in X$. Suppose $H_\kappa \models$ "$\exists y \varphi(y, \phi(x))$". We want $x_0 \in X$ such that

$$H_\kappa \models \text{“}\varphi(\phi(x_0), \phi(x))\text{”}.$$

Since $N_2 \prec H_\kappa$, we have $N_2 \models$ "$\exists y \varphi(y, \phi(x))$". Since $\phi : N_1 \longrightarrow N_2$ is the isomorphism, we have $N_1 \models$ "$\exists y \varphi(y, x)$". Since $N_1 \prec H_\kappa$, we have $H_\kappa \models$ "$\exists y \varphi(y, x)$". Since $X \prec H_\kappa$, we have $x_0 \in X$ such that $H_\kappa \models$ "$\varphi(x_0, x)$". Hence $N_1 \models$ "$\varphi(x_0, x)$". Hence $N_2 \models$ "$\varphi(\phi(x_0), \phi(x))$". Hence $H_\kappa \models$ "$\varphi(\phi(x_0), \phi(x))$".

(3) : $f : (X, \in) \longrightarrow (Y, \in)$ is an isomorphism iff $H_\kappa \models$ "$f : (X, \in) \longrightarrow (Y, \in)$ is an isomorphism".

□

2.5 Lemma. (CH) $P_{AM} = P$ has the ω_2-cc.

Proof. Let $\langle p_i \mid i < \omega_2 \rangle$ be an indexed family of conditions of P_{AM}. Take $N_i \prec H_\kappa$ with $p_i \in N_i$ for each $i < \omega_2$. By CH, we may assume that N_is form a Δ-system with the kernel Δ. We may also assume that $(N_i, \in, p_i)$s are isomorphic. Let $c_{N_i} : N_i \longrightarrow \overline{N_i}$ be the transitive collapse of N_i. By CH again, we may assume that $c_{N_i} \lceil \Delta = c_{N_j} \lceil \Delta$ for all $i < j < \omega_2$. Then $N_i \cap (p_i \cup p_j) = p_i$ and $N_j \cap (p_i \cup p_j) = p_j$. Hence $p_i \cup p_j \cup \{N_i, N_j\} \in P$ and it is a common extension of p_i and p_j in P.

□

2.6 Lemma. (proper) Let $p \in P_{AM} = P$ and $N \prec (H_\kappa, \in, P, \leq, 1, R^P_{\leq})$ be countable with $N \in p$. Then p is (P, N)-generic. By this we mean that for every <u>*predense*</u> subset $A \in N$ of P, $A \cap N$ is <u>*predense*</u> below p.

Proof. Let p, N and A be as in the statement. Let $q \leq p$. We may assume that there exists $a \in A$ with $q \leq a$. Since $q \cap N$ is finite, we may use $q \cap N$ as a parameter in N. We have

$$(H_\kappa, \in, P, \leq, 1, R^P_{\leq}) \models \text{“} \exists q' \text{ in } P \text{ and } \exists a' \in A\, (q \cap N) \subset q',\ q' \leq a' \in A \text{”}.$$

Since $(q \cap N)$, $A \in N \prec H_\kappa$, we may fix q', $a' \in N$. Let

$$r = q \cup \bigcup \{\phi_{NN_1}[q'] \mid N_1 \in q,\ N_1 \cap \omega_1 = N \cap \omega_1\}.$$

Then $r \in P$ and r is a common extension of q, q', and so a' in P. This establishes that $A \cap N$ is predense below p.

□

We consider a situation when isomorphisms get extended, though we see no use of this in this note.

2.7 Lemma. Let $p \in P_{AM} = P$. Let N, $N_1 \prec (H_\kappa, \in, P, \leq, 1, R^P_{\leq})$ such that N, $N_1 \in p$ and $N \cap \omega_1 = N_1 \cap \omega_1$. Let us assume that

$$\phi : (N, \in, P, \leq, 1, R^P_{\leq}) \longrightarrow (N_1, \in, P, \leq, 1, R^P_{\leq}).$$

is the isomorphism. Then ϕ gets extended to

$$\begin{aligned} p \Vdash_P \text{“} \dot{\phi} : (N[\dot{G}], \in, N, \dot{G} \cap N, P \cap N, \leq \cap N, R^P_{\leq} \cap N) \longrightarrow \\ (N_1[\dot{G}], \in, N_1, \dot{G} \cap N_1, P \cap N_1, \leq \cap N_1, R^P_{\leq} \cap N_1) \text{”}. \end{aligned}$$

Proof. We know that p is (P, N)-generic and (P, N_1)-generic. Let G be P-generic over the ground model V with $p \in G$. We argue in $V[G]$. Let us define (abusive notation) $\phi : N[G] \longrightarrow N_1[G]$ by

$$\phi(\tau_G) = \phi(\tau)_G.$$

We have to show several items.

(well-defined): Let τ, $\pi \in N \cap V^P$. Suppose $\tau_G = \pi_G$. We want to show $\phi(\tau)_G = \phi(\pi)_G$. We first note that $N \models$ "τ is a P-name". Hence $N_1 \models$ "$\phi(\tau)$ is a P-name". Hence $\phi(\tau)_G$ gets defined. Next note that if $w \in G \cap N$ and $w \Vdash_P$ "$\tau = \pi$", then $\phi(w) \in G$ and $\phi(w) \Vdash_P$ "$\phi(\tau) = \phi(\pi)$" and so $\phi(\tau)_G = \phi(\pi)_G$ holds. To see this, we argue as follows. Take $q \in G$ such that $q \leq w, p$. Then $N, N_1 \in q$. Since $w \subset N \cap q$, we have $\phi(w) \subset N_1 \cap q \subset q$. Since $\phi(w) \in P$, we conclude $q \leq \phi(w)$. Hence $\phi(w) \in G$.

(one-to-one, onto, and $\in$-homo): Similar to well-definedness.

(extends, N to N_1 homo): Let $x \in N$. Then $\check{x} \in N \cap V^P$ and $\phi(\check{x}) = \check{\phi(x)}$ holds. Hence $\phi(x) = \phi(\check{x})_G$. Since $\phi[N] = N_1$, ϕ is N to N_1 homo.

($G \cap N$ to $G \cap N_1$ homo): Let $g \in G \cap N$. Take $q \in G$ such that $q \leq g, p$. Then $N, N_1 \in q$. Since $g \subset N \cap q$, we have $\phi(g) \subset N_1 \cap q$. Since $\phi(g) \in P$, we have $q \leq \phi(g)$. Hence we conclude $\phi(g) \in N_1 \cap G$. Conversely, let $g \in N_1 \cap G$. Then by considering ϕ^{-1}, we have $\phi^{-1}(g) \in N \cap G$. Hence $g = \phi(\phi^{-1}(g)) \in \phi[N \cap G]$. Hence $\phi[G \cap N] = G \cap N_1$ holds.

□

2.8 Lemma. In the generic extensions V^P, let

$$\dot{\mathcal{E}} = \{N \cap \kappa \mid N \in \bigcup \dot{G}\}.$$

Then $\dot{\mathcal{E}}$ is a mother.

Proof. Let G be P-generic over V. Let $\mathcal{E} = \dot{\mathcal{E}}_G$.

(stationary): In $V[G]$, let $\dot{F}_G = F : [\kappa]^{<\omega} \longrightarrow \kappa$. We want to find $N \cap \kappa \in \mathcal{E} \cap C(F)$, where

$$C(F) = \{X \in [\kappa]^\omega \mid \forall a \in [X]^{<\omega} F(a) \in X\}.$$

To this end take $p \in G$ such that $p \Vdash_P$ "$\dot{F} : [\kappa]^{<\omega} \longrightarrow \kappa$". Take a countable elementary substructure N^* of H_θ, where θ is a sufficiently large regular cardinal, with $H_\kappa, P, \leq, 1, R^P_{=}, p, \dot{F} \in N^*$. Let $N = N^* \cap H_\kappa$ and $q = p \cup \{N\}$. Then $N \prec (H_\kappa, \in, P, \leq, 1, R^P_{=})$, $q \in P$, $N \in q$, and so q is (P, N)-generic. We may assume that $q \in G$. Since $N[G] = N^*[G] \cap H_\kappa^{V[G]}$ and $N^*[G] \cap \kappa \in C(F)$, we have $N \cap \kappa = N[G] \cap \kappa = N^*[G] \cap \kappa \in \mathcal{E} \cap C(F)$.

(1): Let $N \cap \kappa$, $M \cap \kappa \in \mathcal{E}$ such that $N \cap \kappa \neq M \cap \kappa$ and $N \cap \kappa \subset M \cap \kappa$. Take $p \in G$ such that $N, M \in p$. We want $N \cap \omega_1 < M \cap \omega_1$. Suppose $N \cap \omega_1 = M \cap \omega_1$. Then we would have $N \cap \kappa = M \cap \kappa$. Suppose $M \cap \omega_1 < N \cap \omega_1$. Then we would have $N_0 \in p$ such that $M \in N_0$ and $N_0 \cap \omega_1 = N \cap \omega_1$. Then $N \cap \kappa \subset M \cap \kappa \subset N_0 \cap \kappa$, and so $N \cap \kappa = N_0 \cap \kappa$. Hence we would have $N \cap \kappa = M \cap \kappa$. Therefore $N \cap \omega_1 < M \cap \omega_1 < \omega_1$.

(2): Let $N \cap \kappa, N_1 \cap \kappa \in \mathcal{E}$ with $N \cap \omega_1 = N_1 \cap \omega_1$. We want an isomorphism $\phi : (N \cap \kappa, <) \longrightarrow (N_1 \cap \kappa, <)$. Let $p \in G$ with $N, N_1 \in p$. Since $N \cap \omega_1 = N_1 \cap \omega_1$, there exists an isomorphism $\phi : (N, \in, p \cap N) \longrightarrow (N_1, \in, p \cap N_1)$. We check that $\phi[\mathcal{E} \lceil (N \cap \kappa)] = \mathcal{E} \lceil (N_1 \cap \kappa)$. Let $M \cap \kappa \in \mathcal{E} \lceil (N \cap \kappa)$. Then we may assume that $M \in p \cap N$. Hence $\phi[M \cap \kappa] = \phi(M) \cap \kappa \in \mathcal{E} \lceil (N_1 \cap \kappa)$. Conversely, let $M_1 \cap \kappa \in \mathcal{E} \lceil (N_1 \cap \kappa)$. Then we may assume that $M_1 \in p \cap N_1$. Hence $\phi^{-1}(M_1) = M \in p \cap N$. Hence $M_1 \cap \kappa = \phi(M) \cap \kappa = \phi[M \cap \kappa] \in \phi[\mathcal{E} \lceil (N \cap \kappa)]$.

(3): Let $M\cap\kappa$, $N_1\cap\kappa\in\mathcal{E}$ with $M\cap\omega_1<N_1\cap\omega_1$. Let $p\in G$ such that M, $N_1\in p$. Take $N\in p$ such that $M\in N$ and $N\cap\omega_1=N_1\cap\omega_1$. Then $N\cap\kappa\in\mathcal{E}$ such that $M\cap\kappa\in\mathcal{E}\lceil(N\cap\kappa)$.

(4) : Let $M\cap\kappa$, $N_1\cap\kappa\in\mathcal{E}$ with $M\cap\omega_1<N_1\cap\omega_1$ Let $\alpha\in N_1\cap\kappa$ with $\mathrm{cf}(\alpha)\geq\omega_1$. Let $p\in G$ such that M, $N_1\in p$. Take $N\in p$ such that $M\in N$ and $N\cap\omega_1=N_1\cap\omega_1$. Then $M\cap N_1=N\cap\phi(M)$ holds, where $\phi:N\longrightarrow N_1$. Then $M\cap N_1\cap\alpha\subset\phi(M)\cap\alpha\in N_1$. Hence there exists $\rho\in N_1\cap\alpha$ such that $M\cap N_1\cap\alpha\subset\rho$.

□

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