A sequent system for a sublogic of the smallest interpretablity logic

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Abstract. An interpretability logic is an extension of provability logic GL with a binary modal operator \( \triangleright \). The smallest interpretability logic IL is obtained by adding axioms concerning \( \triangleright \) to GL (cf. Visser [Vis97] and Japaridze and de Jongh [JJ98]). The logic IK4 is a sublogic of IL and is obtained by adding the same axioms to normal modal logic K4 as the additional axioms of IL. [Sas02] gave a cut-free sequent system for IK4 (see also [Sas01]). Here we give another cut-free sequent system for IK4. Both of the system in [Sas02] and the system here satisfy kinds of subformula property, however, our new system has nicer one. In the system in [Sas02], a formula \( B \triangleright D \) possibly occurs in a cut-free proof figure for \( A \triangleright B \rightarrow C \triangleright D \), while in the new system doesn’t.

1 Preliminaries

The language of the logic IK4 contains two modal operators \( \Box \) and \( \triangleright \). However, we can show the equivalence between \( \Box A \) and \( (A \supset \bot) \triangleright \bot \). Hence, we do not have to treat \( \Box \) as a primary operator. Systems for interpretability logics with two primary modal operators are much more complicated than the ones with one primary modal operator. So, we treat \( \Box A \) as an abbreviation of \( (A \supset \bot) \triangleright \bot \).

Definition 1.1. The set WFF of formulas are defined inductively as follows.

(1) a propositional variable belongs to WFF,
(2) \( \bot \in \text{WFF} \),
(3) \( A, B \in \text{WFF} \) implies \( (A \land B), (A \lor B), (A \supset B), (A \triangleright B) \in \text{WFF} \).

An element of WFF is said to be a formula, especially a formula of the form \( A \triangleright B \) is said to be a \( \triangleright \)-formula. The expressions \( \neg A \), \( \Box A \) and \( \Diamond A \) are abbreviations for \( A \supset \bot \), \( \Box \diamondsuit \bot \) and \( \neg (A \triangleright \bot) \), respectively. By IK4, we mean the smallest set of formulas containing all the tautologies and axioms

\[
\begin{align*}
(K) & : \Box(p \supset q) \supset (\Box p \supset \Box q), \\
(4) & : \Box p \supset \Box \Box p, \\
(J1) & : \Box(p \supset q) \supset (p \triangleright q), \\
(J2) & : (p \triangleright q) \land (q \triangleright r) \supset (p \triangleright r), \\
(J3) & : (p \triangleright r) \land (q \triangleright r) \supset ((p \lor q) \triangleright r), \\
(J5) & : (\Box p) \triangleright p,
\end{align*}
\]

and closed under modus ponens, substitution and necessitation. The axiom \((J4)\) is necessary of the logic in the language with \( \Box \) as a primary one, while it does not necessary of the logic in our language. So, we do not need \((J4)\).

To introduce a sequent system, we need some preparations. We use Greek letters, possibly with suffixes, for finite sets of formulas, especially we use \( \Sigma \), possibly with suffixes, for finite sets of \( \triangleright \)-formulas. For each prefix \( \odot \in \{\Box, \Diamond, \neg\} \), the expression \( \odot \Gamma \) denotes the set \( \{\odot A \mid A \in \Gamma\} \). Similarly, \( \Gamma \triangleright \bot \) denotes \( \{A \triangleright \bot \mid A \in \Gamma\} \). By a sequent, we mean the expression

\[ \Gamma \rightarrow \Delta. \]

For brevity’s sake, we write

\[ A_1, \ldots, A_k, \Gamma_1, \ldots, \Gamma_\ell \rightarrow \Delta_1, \ldots, \Delta_m, B_1, \ldots, B_n \]

instead of

\[ \{A_1, \ldots, A_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell \rightarrow \Delta_1 \cup \cdots \cup \Delta_m \cup \{B_1, \ldots, B_n\}. \]
We put
\[ \text{an}(\{A\} \rightarrow \Delta) = A \text{ (antecedent)}. \]
\[ \text{Su}(\Gamma \rightarrow \Delta) = \Delta \text{ (succeident)}. \]

**Definition 1.2.** Let
\[ S_1 \cdots S_n \]
be a sequence of \( n \) sequents. The depth \( d_{S_1} \cdots S_n(o_i) \) of the occurrence \( o_i \) of \( S_i \) denotes \( n + 1 - i \).

We use \( \text{SEQ} \), possibly with suffixes, for sequences of sequents. Let \( \text{SEQ} \) be a sequence of sequents. For an occurrence \( o \) of a sequent \( S \) in \( \text{SEQ} \), we also use the expressions \( \text{an}(o) \) and \( \text{Su}(o) \) for \( \text{an}(S) \) and \( \text{Su}(S) \), respectively.

**Definition 1.3.** A sequent is said to be on a sequent \( \Sigma \rightarrow A \triangleright B \) if it is of the form
\[ C \rightarrow \Delta, \]
where \( C \in \{A\} \cup \{E \mid D 	riangleright E \in \Sigma\} \) and \( B \in \Delta \subseteq \{D \mid D 	riangleright E \in \Sigma\} \cup \{B\} \).

**Remark 1.4.**
(1) There are finitely many sequents on \( \Sigma \rightarrow A \triangleright B \). If the number of formulas in \( \Sigma \) is \( n \), then the number of sequents on \( \Sigma \rightarrow A \triangleright B \) is less than or equal to \( (n + 1) \times 2^n \).
(2) If \( S \) is a sequent on \( \Sigma \rightarrow A \triangleright B \), then it is also on \( \Sigma \cup \Sigma' \rightarrow A \triangleright B \).

**Definition 1.5.** We say that a sequence \( \text{SEQ} \) of sequents is on a sequent \( \Sigma \rightarrow A \triangleright B \) if each sequent in \( \text{SEQ} \) is on \( \Sigma \rightarrow A \triangleright B \) and \( \text{an}(S) = A \) for some sequent \( S \) in \( \text{SEQ} \).

**Definition 1.6.** Let \( \text{SEQ} \) be a sequence on \( \Sigma \rightarrow A \triangleright B \) and let \( \Sigma' \) be a subset of \( \Sigma \). Let \( \text{OCC} \) be the set of occurrences of sequents in \( \text{SEQ} \).
(1) A mapping
\[ f : \{ (o, C) \mid o \in \text{OCC}, C \in \text{Su}(o)_B \} \rightarrow \text{OCC} \]
is called a mapping on \( \text{SEQ}/\Sigma' \) if \( C \triangleright \text{an}(f(o, C)) \in \Sigma' \) and \( d_{\text{SEQ}}(o) > d_{\text{SEQ}}(f(o, C)) \) for any occurrence \( o \) and any formula \( C \in \text{Su}(o)_B \). By \( \text{Im}(f) \), we mean the range of \( f \).
(2) A mapping
\[ f : \{ (o, C) \mid o \in \text{OCC}, C \in \text{Su}(o)_B \} \rightarrow \text{OCC} \]
is called a pseudo-onto mapping on \( \text{SEQ}/\Sigma' \rightarrow A \triangleright B \) if \( f \) is on \( \text{SEQ}/\Sigma' \) and
\[ \{ o \mid o \in \text{OCC}, \text{an}(o) \neq A \} \subseteq \text{Im}(f). \]

**Definition 1.7.** We say that a sequence \( \text{SEQ} \) is essential on a sequent \( \Sigma \rightarrow A \triangleright B \) if the following three conditions hold:
(1) \( \text{SEQ} \) is on \( \Sigma \rightarrow A \triangleright B \),
(2) there exists an pseudo-onto mapping on \( \text{SEQ}/\Sigma \rightarrow A \triangleright B \),
(3) \( \text{an}(o_1) \neq \text{an}(o_2) \) for any different occurrences \( o_1 \) and \( o_2 \) in \( \text{SEQ} \).

**Corollary 1.8.** Let \( n \) be the number of elements in \( \Sigma \).
(1) An essential sequence on \( \Sigma \rightarrow A \triangleright B \) has at most \( n + 1 \) sequents.
(2) The number of essential sequences on \( \Sigma \rightarrow A \triangleright B \) is less than \( (n + 1)! \times 2^{n \times (n + 1)} \).

Proof. (1) is from Definition 1.7(3), and (2) from Remark 1.4(1). \( \square \)
Lemma 1.9. Let $S_1 \cdots S_n$ be an essential sequence on $\Sigma \rightarrow A \triangleright B$. Then

1. $\text{Su}(S_n) = \{ B \}$,
2. $\text{an}(S_1) = A$,
3. $\text{an}(S_i) = A$ if and only if $i = 1$.

Proof. Let $o_i$ be the occurrence of $S_i$. Then

$$n \geq d_{S_1} \cdots s_n(o_i) \geq 1 \cdot \cdot \cdot (\ast 1)$$

Since the sequence is essential, there exists a pseudo-onto mapping $f$ on $$(S_1 \cdots S_n)/\Sigma \rightarrow A \triangleright B.$$

For (1): Suppose that $\text{Su}(S_n) \neq \{ B \}$. Since $S_n$ is on $\Sigma \rightarrow A \triangleright B$, we have $\{ B \} \subseteq \text{Su}(S_n)$, and so, $D \in \text{Su}(S_n)B$ for some $D$. Using Definition 1.6, $1 = d_{S_1} \cdots s_n(o_n) > d_{S_1} \cdots s_n(f(o_n, D))$. This is in contradiction with ($\ast 1$).

For (2): Suppose that $\text{an}(S_1) \neq A$. Since $f$ is pseudo-onto, there exit an occurrence $o'$ and a formula $D \in \text{Su}(o')B$ such that $o_1 = f(o', D)$. Also we have $d_{S_1} \cdots s_n(o') > d_{S_1} \cdots s_n(f(o', D)) = d_{S_1} \cdots s_n(o_1) = n$. This is in contradiction with ($\ast 1$).

For (3): From (2), we have “if” part. We show “only if” part. Suppose that $\text{an}(S_i) = A$. By (2) and Definition 1.7(3), we have $o_i = o_1$. Hence $i = 1$.

Example 1.10. Here we show some examples of essential sequences.

1. A sequence of a sequent $p \rightarrow q$

is only one essential sequence on $\rightarrow p \triangleright q$.

2. Two sequences of sequents $p \rightarrow q, p_1 \quad q_1 \rightarrow q$

and

$p \rightarrow q$

are essential on $p_1 \triangleright q_1 \rightarrow p \triangleright q$.

3. Two sequences $p \rightarrow q, p_1 \quad q_1 \rightarrow q, p_2 \quad q_2 \rightarrow q$

and

$p \rightarrow q, p_1, p_2 \quad q_1 \rightarrow q \quad q_2 \rightarrow q$

are essential on $p_1 \triangleright q_1, p_2 \triangleright q_2 \rightarrow p \triangleright q$.

4. A sequence $p \triangleright \bot \rightarrow p, q \quad q \triangleright \bot \rightarrow p, q \quad \bot \rightarrow p$

is essential on $q \triangleright (q \triangleright \bot), q \triangleright \bot \rightarrow (p \triangleright \bot) \triangleright p$.

5. A sequence $p \rightarrow q, \neg r \quad \bot \rightarrow q$

is essential on $\square r \rightarrow p \triangleright q$. 

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(6) A sequence
\[ \neg \Box p \rightarrow \bot, \neg p \rightarrow \bot \]
is essential on \[ \Box p \rightarrow \Box \Box p. \]

**Definition 1.11.** Let \( S, S_1, \cdots, S_n \) be sequents.
(1) By \( S^* \), we mean the sequent
\[ \Gamma, \Delta \vdash \bot \rightarrow \Delta \]
if \( S \) is \( \Gamma \rightarrow \Delta \).
(2) By \( \text{SEQ}^* \), we mean the sequence
\[ S_1^* \cdots S_n^* \]
if \( \text{SEQ} \) is
\[ S_1 \cdots S_n. \]

Our system \( \text{GIK}4 \) is defined from the following axioms and inference rules in the usual way.

**Axioms of GIK4**
\[
A \rightarrow A \\
\bot \rightarrow 
\]

**Inference rules of GIK4**
\[
\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} (T \rightarrow) \\
\frac{\Gamma \rightarrow \Delta, A}{\Gamma, \Pi \rightarrow \Lambda} (\text{cut}) \\
\frac{A_1, \Gamma \rightarrow \Delta}{A_1 \land A_2, \Gamma \rightarrow \Delta} (\land \rightarrow) \\
\frac{A, \Gamma \rightarrow \Delta}{A \lor B, \Gamma \rightarrow \Delta} (\lor \rightarrow) \\
\frac{\Gamma \rightarrow \Delta, A \lor B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta} (\supset \rightarrow) \\
\frac{\text{SEQ}^*}{\Sigma \rightarrow A \supset B} (\supset \text{IK}4)
\]
where \( \text{SEQ} \) is essential on \( \Sigma \rightarrow A \supset B. \)

2 Equivalence between IK4 and GIK4

The main theorem in this section is

**Theorem 2.1.** \( A \in \text{IK}4 \) if and only if \( \rightarrow A \in \text{GIK}4 \).

To prove the theorem above, we provide some preparations.

By \( \text{GK}4 \), we mean the system obtained from \( \text{GIK}4 \) by replacing \( (\supset \text{IK}4) \) by
\[
\frac{\Gamma, \Box \Gamma \rightarrow A}{\Box \Gamma \rightarrow \Box A} (\Box \text{K}4).
\]
By $\text{GK4} + J$, we mean the system obtained from $\text{GK4}$ by adding following four axioms:

1. $(GJ1) : \Box(A \supset B) \rightarrow A \supset B,$
2. $(GJ2) : A \supset B, B \supset C \rightarrow A \supset C,$
3. $(GJ3) : A \supset C, B \supset C \rightarrow (A \lor B) \supset C,$
4. $(GJ5) : \rightarrow \Diamond A \supset A.$

It is known that $\text{GK4}$ enjoys cut-elimination theorem and that $\rightarrow A \in \text{GK4}$ if and only if $A \in \text{K4}.$ So, we have

Lemma 2.2. $A \in \text{IK4}$ if and only if $\rightarrow A \in \text{GK4} + J.$

Lemma 2.3. $A \in \text{GK4} + J$ implies $\rightarrow A \in \text{GIK4}.$

Proof. It is sufficient to show that four axioms $(J1), (J2), (J3)$ and $(J5)$ are provable in $\text{GIK4}$ and $(\Box_{I\text{K4}})$ holds in $\text{GIK4}.$

For $(J1)$: We note that the sequence

$$A \rightarrow B, \neg(A \supset B) \quad \perp \rightarrow B$$

is essential on $\Box(A \supset B) \rightarrow A \supset B.$ On the other hand, it is easily seen that the following sequents are provable in $\text{GIK4}:$

$$A, B \supset \perp, \Box(A \supset B) \rightarrow B, \neg(A \supset B),$$

$$\perp, B \supset \perp \rightarrow B.$$ 

So, using $(\supset_{I\text{K4}}),$ $(J1)$ is provable in $\text{GIK4}.$

The other axioms and rules can be shown by the following $(\supset_{I\text{K4}})$ in a similar way.

$$A, C \supset \perp, A \supset \perp \rightarrow C, A \quad B, C \supset \perp, B \supset \perp \rightarrow C, B \quad C, C \supset \perp \rightarrow C$$

$$A \supset B, B \supset C \rightarrow A \supset C$$

$$A \lor B, C \supset \perp, A \supset \perp, B \supset \perp \rightarrow C, A, B \quad C, C \supset \perp \rightarrow C$$

$$A \supset C, B \supset C \rightarrow A \lor B \supset C$$

$$\Box A, A \supset \perp \rightarrow A$$

$$\rightarrow \Box A \supset A$$

$$\neg A, \Box \Gamma, \perp \supset \perp \rightarrow \neg \Gamma, \perp$$

$$\perp, \perp \supset \perp \rightarrow \perp$$

$$\Box \Gamma \rightarrow \Box A$$

Lemma 2.4. The following rules hold in $\text{GK4} + J.$

1. $(1) \quad \Gamma \rightarrow \Box(A \supset B) \in \text{GK4} + J$ implies $\Gamma \rightarrow A \supset B \in \text{GK4} + J,$
2. $(2) \quad$ if $\Gamma \rightarrow A \supset B$ and $\Gamma \rightarrow B \supset C$ are provable in $\text{GK4} + J,$ then so is $\Gamma \rightarrow A \supset C,$
3. $(3) \quad$ if $\Gamma \rightarrow A \supset C$ and $\Gamma \rightarrow B \supset C$ are provable in $\text{GK4} + J,$ then so is $\Gamma \rightarrow A \lor B \supset C,$
4. $(4) \quad$ if $(B \lor \Diamond B) \supset B \in \text{GK4} + J,$
5. $(5) \quad$ if $\Gamma \rightarrow (B \lor \Box B) \in \text{GK4} + J$ implies $\Gamma \rightarrow A \supset B \in \text{GK4} + J,$
6. $(6) \quad$ if $\Gamma \rightarrow A \supset B \in \text{GK4} + J$ implies $\Gamma \rightarrow (A \lor \Box A) \supset B \in \text{GK4} + J.$

Proof. We obtain (1), (2) and (3), from $(GJ1), (GJ2)$ and $(GJ3),$ respectively. (5) and (6) are from (2) and (4).

So, we only show (4). By $(GJ1),$ it is easily seen that $\rightarrow B \supset B \in \text{GK4} + J.$ Using $(GJ5)$ and (3), we obtain (4).

Lemma 2.5. Let $\text{SEQ}$ be an essential sequence on $\Sigma \rightarrow A \supset B$ such that each sequent occurring in $\text{SEQ}^*$ is provable in $\text{GK4} + J$ and let $o$ be an occurrence of a sequent in $\text{SEQ}.$ Then

$$\Sigma \rightarrow \text{an}(o) \supset B \in \text{GK4} + J.$$
Proof. The sequent occurring at $o$ is of the form

$$\text{an}(o) \rightarrow \text{Su}(o),$$

where $B \in \text{Su}(o)$, and so,

$$\text{an}(o), \text{Su}(o) \triangleright \bot \rightarrow \text{Su}(o)$$

occurs in $\text{SEQ}'$ and is provable in $\text{GIK}4 + J$. Remember $\Diamond D = \neg (D \triangleright \bot)$ (cf. Notation 6.1.2), and we have

$$\text{an}(o) \rightarrow \bigvee_{D \in \text{Su}(o)} (D \vee \Diamond D) \in \text{GK}4 + J.$$ 

Using ($\rightarrow \Box$), ($\Box K_4$)

$$\rightarrow \Box \text{an}(o) \triangleright \bigvee_{D \in \text{Su}(o)} (D \vee \Diamond D)) \in \text{GK}4 + J.$$ 

Using Lemma 2.4(1),

$$\rightarrow \text{an}(o) \triangleright (\bigvee_{D \in \text{Su}(o)} (D \vee \Diamond D)) \in \text{GK}4 + J \cdots (*1)$$

Now, we use an induction on $d_{\text{SEQ}}(o)$.

If $d_{\text{SEQ}}(o) = 1$, then $o$ occurs at the end of $\text{SEQ}$. By Lemma 1.9(1), we have $\text{Su}(o) = \{B\}$. Using ($*1$), we have

$$\rightarrow \text{an}(o) \triangleright (B \vee \Diamond B) \in \text{GK}4 + J.$$ 

Using Lemma 2.4(5),

$$\rightarrow \text{an}(o) \triangleright B \in \text{GK}4 + J.$$ 

Using ($T \rightarrow$), possibly several times, we obtain the lemma.

Suppose that $d_{\text{SEQ}}(o) > 1$ and the lemma holds for any occurrence $o'$ of a sequent such that $d_{\text{SEQ}}(o') < d_{\text{SEQ}}(o)$. Let $C$ be a formula in $\text{Su}(o)_B$. Since $\text{SEQ}$ is essential, there exists a pseudo-onto mapping $f$ on $\text{SEQ}/\Sigma \rightarrow A \triangleright B$ that satisfies

$$C \triangleright \text{an}(f(o,C)) \in \Sigma \text{ and } d_{\text{SEQ}}(f(o,C)) < d_{\text{SEQ}}(o).$$

By $C \triangleright \text{an}(f(o,C)) \in \Sigma$, we have

$$\Sigma \rightarrow C \triangleright \text{an}(f(o,C)) \in \text{GK}4 + J.$$ 

Also by $d_{\text{SEQ}}(f(o,C)) < d_{\text{SEQ}}(o)$ and the induction hypothesis,

$$\Sigma \rightarrow \text{an}(f(o,C)) \triangleright B \in \text{GK}4 + J.$$ 

Using Lemma 2.4(2),

$$\Sigma \rightarrow C \triangleright B \in \text{GK}4 + J.$$ 

Using Lemma 2.4(6),

$$\Sigma \rightarrow (C \vee \Diamond C) \triangleright B \in \text{GK}4 + J.$$ 

By Lemma 2.4(4), we also have

$$\Sigma \rightarrow (B \vee \Diamond B) \triangleright B \in \text{GK}4 + J.$$ 

Using Lemma 2.4(3), possibly several times,

$$\Sigma \rightarrow (\bigvee_{D \in \text{Su}(o)} (D \vee \Diamond D)) \triangleright B \in \text{GK}4 + J.$$ 

Using ($*1$) and Lemma 2.4(2),

$$\Sigma \rightarrow \text{an}(o) \triangleright B \in \text{GK}4 + J.$$
Corollary 2.6. Let \( \text{SEQ} \) be an essential sequence on \( \Sigma \rightarrow A \triangleright B \) such that each sequent occurring in \( \text{SEQ}^* \) is provable in \( \text{GK}4 + J \). Then
\[
\Sigma \rightarrow A \triangleright B \in \text{GK}4 + J.
\]

Proof. Since \( \text{SEQ} \) is on \( \Sigma \rightarrow A \triangleright B \), there exists an occurrence \( o \) of a sequent in \( \text{SEQ} \) such that \( \text{an}(o) = A \). So, using Lemma 2.5, we obtain the corollary.

Lemma 2.7. \( \rightarrow A \in \text{GIK}4 \) implies \( \rightarrow A \in \text{GK}4 + J \).

Proof. From Corollary 2.6, we can see that the inference rule \( (\triangleright_{IK}4) \) holds in \( \text{GIK}4 + J \). Hence, we obtain the lemma.

From Lemma 2.3, Lemma 2.2 and Lemma 2.7, we obtain Theorem 2.1.

3 Cut-elimination theorem for GIK4

In this section, we prove cut-elimination theorem for GIK4.

Theorem 3.1. If \( \Gamma \rightarrow \Delta \in \text{GIK}4 \), then there exists a cut-free proof figure for \( \Gamma \rightarrow \Delta \) in \( \text{GIK}4 \).

It is easily seen that Theorem 3.1 follows from the following lemma.

Lemma 3.2. Let \( P \) be a proof figure satisfying the following two conditions:
(1) the inference rule \( I \) introducing the end sequent of \( P \) is a cut,
(2) \( I \) is the only one cut in \( P \),
Then there exists a cut-free proof figure for the end sequent of \( P \).

To prove the lemma above, we use the usual way, which originated with Gentzen [Gen35]. We have only to show the case concerning the operator \( \triangleright \) since the other cases can be shown in the usual way. In order to show the case, we provide some preparations.

Lemma 3.3. Let \( \text{SEQ} \) be a sequence on \( \Sigma \rightarrow A \triangleright B \) and let it be that \( \Sigma' \subseteq \Sigma \). If there exists a pseudo-onto mapping \( f \) on \( \text{SEQ} / \Sigma' \rightarrow A \triangleright B \), then \( \text{SEQ} \) is also on \( \Sigma \rightarrow A \triangleright B \).

Proof. By Definition 1.5, it is sufficient to show the following two:
(1) each sequent \( S \) in \( \text{SEQ} \) is on \( \Sigma' \rightarrow A \triangleright B \),
(2) \( \text{an}(S) = A \) for some sequent \( S \) in \( \text{SEQ} \).

Immediately, we obtain (2) since \( \text{SEQ} \) is on \( \Sigma \rightarrow A \triangleright B \). So, we show (1). Let \( o \) be an occurrence of \( S \). Since \( S \) is on \( \Sigma \rightarrow A \triangleright B \), \( B \in \text{Su}(o) = \text{Su}(S) \). Then we have only to show the following two:
(1.1) \( \text{Su}(o)_B = \text{Su}(S)_B \subseteq \{ C \mid C \triangleright D \in \Sigma' \} \).
(1.2) \( \text{an}(o) = \text{an}(S) \in \{ A \} \cup \{ D \mid C \triangleright D \in \Sigma' \} \).

For (1.1): Let \( E \) be a formula in \( \text{Su}(o)_B \). Since \( f \) is on \( \text{SEQ} / \Sigma' \), \( E \triangleright \text{an}(f(o, E)) \in \Sigma' \). Hence, we obtain (1.1).

For (1.2): If \( \text{an}(o) = A \), then (1.2) is trivial. Let it be that \( \text{an}(o) \neq A \). Since \( f \) is pseudo-onto, there exist an occurrence \( o' \) and a formula \( E \in \text{Su}(o')_B \) such that \( f(o', E) = o \). Since \( f \) is on \( \text{SEQ} / \Sigma' \), we have \( E \triangleright \text{an}(o) = E \triangleright \text{an}(f(o', E)) \in \Sigma' \).

Hence, we obtain (1.2).
A sequence obtained from a sequence $\text{SEQ}$ of sequents by removing some occurrences of sequents, or $\text{SEQ}$ itself, is called a \textit{subsequence} of $\text{SEQ}$.

Let $\text{SEQ}$ and $\text{SEQ'}$ be sequences of sequents, possibly empty, and let $o$ be an occurrence of a sequent in $\text{SEQ}$. Let $\text{SEQ''}$ be the sequence obtained from $\text{SEQ}$ by replacing $o$ by $\text{SEQ'}$. If an occurrence $o_1$ occurs in $\text{SEQ}$ and is different from $o$, there exists the corresponding occurrence in $\text{SEQ''}$ and the sequent at the occurrence in $\text{SEQ''}$ also occurs at $o_1$. So, we use the same expression $o_1$ for the corresponding occurrence in $\text{SEQ''}$. Similarly, we use the same expression for an occurrence $o'$ in $\text{SEQ'}$ and for its corresponding occurrence in $\text{SEQ''}$.

\textbf{Remark 3.4.} Let $\text{SEQ}$ and $\text{SEQ'}$ be a sequences of sequents, possibly empty, and let $o$ be an occurrence of a sequent in $\text{SEQ}$. Let $\text{SEQ''}$ be the sequence obtained from $\text{SEQ}$ by replacing $o$ by $\text{SEQ'}$. Then for any occurrences $o_1$ and $o_2$ in $\text{SEQ''}$, we have the following:

1. The case that neither $o_1$ nor $o_2$ occurs in $\text{SEQ'}$:
   \[ d_{\text{SEQ}}(o_1) > d_{\text{SEQ}}(o_2) \text{ if and only if } d_{\text{SEQ'}}(o_1) > d_{\text{SEQ'}}(o_2). \]

2. The case that $o_2$ occurs in $\text{SEQ'}$ but $o_1$ does not:
   \[ d_{\text{SEQ}}(o_1) > d_{\text{SEQ}}(o_2) \text{ if and only if } d_{\text{SEQ'}}(o_1) > d_{\text{SEQ'}}(o_2). \]

3. The case that $o_1$ occurs in $\text{SEQ'}$ but $o_2$ does not:
   \[ d_{\text{SEQ}}(o) > d_{\text{SEQ}}(o_2) \text{ if and only if } d_{\text{SEQ'}}(o) > d_{\text{SEQ'}}(o_2). \]

4. The case that both of $o_1$ and $o_2$ occur in $\text{SEQ'}$:
   \[ d_{\text{SEQ}}(o_1) > d_{\text{SEQ}}(o_2) \text{ if and only if } d_{\text{SEQ'}}(o_1) > d_{\text{SEQ'}}(o_2). \]

\textbf{Lemma 3.5.} Let $\text{SEQ}$ be a sequence on $\Sigma \rightarrow A \triangleright B$ and let it be that $\Sigma' \subseteq \Sigma$. If there exists a mapping $f$ on $\text{SEQ}/\Sigma'$, then there exist a subsequence $\text{SEQ'}$ of $\text{SEQ}$ on $\Sigma \rightarrow A \triangleright B$ and a pseudo-onto mapping on $\text{SEQ'}/\Sigma' \rightarrow A \triangleright B$.

Proof. We put
\[ \#(\text{SEQ}, f) = \max\{|d_{\text{SEQ}}(o)| o \notin \text{Im}(f) \text{ and } \text{an}(o) \neq A\} \cup \{0\} \]
and we use an induction on $\#(\text{SEQ}, f)$.

If $\#(\text{SEQ}, f) = 0$, then $f$ is a pseudo-onto mapping on $\text{SEQ}/\Sigma' \rightarrow A \triangleright B$ and $\text{SEQ}$ is a subsequence of $\text{SEQ}$ on $\Sigma \rightarrow A \triangleright B$.

Suppose that $\#(\text{SEQ}, f) > 0$ and the lemma holds for any pair $(\text{SEQ'}, f')$ such that $\#(\text{SEQ'}, f') < \#(\text{SEQ}, f)$. Then there exists an occurrence $o \in \{o \mid o \notin \text{Im}(f) \text{ and } \text{an}(o) \neq A\}$ such that
\[ \#(\text{SEQ}, f) = d_{\text{SEQ}}(o). \]

We note that for any $o'$ in $\text{SEQ}$,
\[ d_{\text{SEQ}}(o') > d_{\text{SEQ}}(o) \text{ implies } o' \notin \{o \mid o \notin \text{Im}(f) \text{ and } \text{an}(o) \neq A\}. \cdots (\ast 1) \]

Let $\text{SEQ'}$ be the sequence obtained from $\text{SEQ}$ by removing $o$ and let $f'$ be a mapping obtained from $f$ by restricting its domain into the set of pairs for $\text{SEQ'}$. We show the following three:

1. $\text{SEQ'}$ is on $\Sigma \rightarrow A \triangleright B$,
2. $f'$ is a mapping on $\text{SEQ'}/\Sigma'$,
3. $\#(\text{SEQ'}, f') < d_{\text{SEQ}}(o) = \#(\text{SEQ}, f)$.

By these conditions and the induction hypothesis, there exist a subsequence $\text{SEQ''}$ of $\text{SEQ'}$ on $\Sigma \rightarrow A \triangleright B$ and a pseudo-onto mapping on $\text{SEQ''}/\Sigma' \rightarrow A \triangleright B$. Since $\text{SEQ'}$ is a subsequence of $\text{SEQ}$, so is $\text{SEQ''}$, and so, we can obtain the lemma.
For (1): It is sufficient to show the following two:

(1.1) each sequent in $\text{SEQ}'$ is on $\Sigma \rightarrow A \triangleright B$,

(1.2) $\text{an}(S) = A$ for some sequent $S$ in $\text{SEQ}$.

We show (1.1). Let $S$ be a sequent in $\text{SEQ}'$. Since $\text{SEQ}'$ is a subsequence of $\text{SEQ}$, $S$ also occurs in $\text{SEQ}$. Since $\text{SEQ}$ is on $\Sigma \rightarrow A \triangleright B$, $S$ is on $\Sigma \rightarrow A \triangleright B$. Hence we obtain (1.1).

We show (1.2). Since $\text{SEQ}$ is on $\Sigma \rightarrow A \triangleright B$, we have $\text{an}(S) = A$ for some $S$ in $\text{SEQ}$. Using $\text{an}(o) \neq A$, $S$ does not occur at $o$, and so, $S$ occurs in $\text{SEQ}'$. Hence we obtain (1.2).

For (2): Note that $f'$ does not have to be pseudo-onto. So, by $o \notin \text{Im}(f)$, (2) can be shown.

For (3): It is sufficient to show that for any occurrence $o'$ in $\text{SEQ}'$,

$$d_{\text{SEQ}'}(o') \geq d_{\text{SEQ}}(o) \text{ implies } o' \notin \{o \mid o \notin \text{Im}(f') \text{ and } \text{an}(o) \neq A\}.$$ 

In other words,

$$d_{\text{SEQ}'}(o') \geq d_{\text{SEQ}}(o) \text{ implies either } o' \in \text{Im}(f') \text{ or } \text{an}(o') = A.$$ 

Let $o'$ be an occurrence in $\text{SEQ}'$ and let it be that

$$d_{\text{SEQ}'}(o') \geq d_{\text{SEQ}}(o).$$

We note that $o'$ also occurs in $\text{SEQ}$ and

$$d_{\text{SEQ}}(o') = d_{\text{SEQ}'}(o') + 1.$$ 

Hence

$$d_{\text{SEQ}}(o') = d_{\text{SEQ}'}(o') + 1 \geq d_{\text{SEQ}}(o) + 1,$$

and so,

$$d_{\text{SEQ}}(o') > d_{\text{SEQ}}(o).$$

Using (*1),

$$o' \notin \{o \mid o \notin \text{Im}(f) \text{ and } \text{an}(o) \neq A\}.$$ 

In other words, either $o' \in \text{Im}(f)$ or $\text{an}(o) = A$. If $\text{an}(o) = A$, then we obtain (3). So, we assume that $o' \notin \text{Im}(f)$. Then there exist $o''$ in $\text{SEQ}$ and $E \in \text{Su}(o'')B$ such that $f(o'', E) = o'$. Since $f$ is on $\text{SEQ}/\Sigma'$, we have

$$d_{\text{SEQ}}(o'') > d_{\text{SEQ}}(f(o'', E)) = d_{\text{SEQ}}(o') > d_{\text{SEQ}}(o),$$

and so, $o''$ also occurs in $\text{SEQ}'$. Hence $f'(o'', E) = o'$, i.e., $o' \in \text{Im}(f')$, we obtain (3).

Lemma 3.6. Let $\text{SEQ}$ be a sequence on $\Sigma \rightarrow A \triangleright B$ and let $f$ be a pseudo-onto mapping on $\text{SEQ}/\Sigma \rightarrow A \triangleright B$. Then there exists an essential subsequence $\text{SEQ}'$ of $\text{SEQ}$ on $\Sigma \rightarrow A \triangleright B$.

Proof. We use an induction of the number $\#(\text{SEQ})$ of pairs $(o_1, o_2)$ of different occurrences in $\text{SEQ}$ such that $\text{an}(o_1) = \text{an}(o_2)$.

If $\#(\text{SEQ}) = 0$, then $\text{SEQ}$ is an essential subsequence of $\text{SEQ}$.

Suppose that $\#(\text{SEQ}) > 0$ and the lemma holds for any $\text{SEQ}'$ such that $\#(\text{SEQ}') < \#(\text{SEQ})$. Then there exists a pair $(o_1, o_2)$ of different occurrences of sequents such that $\text{an}(o_1) = \text{an}(o_2)$. Without loss of generality, we assume that $d_{\text{SEQ}}(o_1) > d_{\text{SEQ}}(o_2)$.

Let $\text{SEQ}'$ be the sequence obtained from $\text{SEQ}$ by removing $o_2$. We show the following three:

(1) $\#(\text{SEQ}') < \#(\text{SEQ})$,

(2) $\text{SEQ}'$ is on $\Sigma \rightarrow A \triangleright B$,

(3) there exists a mapping $f'$ on $\text{SEQ}'/\Sigma$.

(1) is trivial. We show (2). Since $\text{SEQ}'$ is a subsequence of $\text{SEQ}$, we have only to show that $\text{an}(o') = A$ for some occurrence $o'$ in $\text{SEQ}'$. Since $\text{SEQ}$ is on the sequent $\Sigma \rightarrow A \triangleright B$, $\text{an}(o) = A$ for some occurrence $o$ in $\text{SEQ}$. If $o = o_1$, then $A = \text{an}(o) = \text{an}(o_1) = \text{an}(o_2)$ and $o_2$ occurs in $\text{SEQ}'$; if not, $o$ occurs in $\text{SEQ}'$.

We show (3). We define a mapping $f'$ on $\text{SEQ}'/\Sigma$. A mapping $f'$ is defined as follows.

$$f'(o, E) = \begin{cases} f(o, E) & \text{if } f(o, E) \neq o_1 \\ o_2 & \text{if } f(o, E) = o_1 \end{cases}$$
In order to show that \( f' \) on \( \text{SEQ}'/\Sigma \), it is sufficient to show \( E \triangleright \text{an}(f'(o, E)) \in \Sigma \) and \( d_{\text{SEQ}}(o) > d_{\text{SEQ}}(f(o, E)) \).

If \( f(o, E) \neq o_1 \), then
\[
E \triangleright \text{an}(f'(o, E)) = E \triangleright \text{an}(f(o, E)) \in \Sigma;
\]
if not,
\[
E \triangleright \text{an}(f'(o, E)) = E \triangleright \text{an}(o_2) = E \triangleright \text{an}(o_1) = E \triangleright \text{an}(f(o, X)) \in \Sigma.
\]

If \( f(o, E) \neq o_1 \), then
\[
d_{\text{SEQ}}(o) > d_{\text{SEQ}}(f(o, E)) = d_{\text{SEQ}}(f'(o, E));
\]
if not,
\[
d_{\text{SEQ}}(o) > d_{\text{SEQ}}(f(o, E)) = d_{\text{SEQ}}(o_1) > d_{\text{SEQ}}(o_2) = d_{\text{SEQ}}(f'(o, E)).
\]
Hence in any case, \( d_{\text{SEQ}}(o) > d_{\text{SEQ}}(f'(o, E)) \), and so, we have \( d_{\text{SEQ}}(o) > d_{\text{SEQ}}(f'(o, E)) \).

Hence \( f' \) is on \( \text{SEQ}'/\Sigma \).

By (2),(3) and Lemma 3.5, there exist a subsequence \( \text{SEQ}'' \) of \( \text{SEQ}' \) on \( \Sigma \rightarrow A \triangleright B \) and a pseudo-onto mapping \( f'' \) on \( \text{SEQ}''/\Sigma \rightarrow A \triangleright B \). Since \( \text{SEQ}'' \) is a subsequence of \( \text{SEQ}' \), we have \( \#(\text{SEQ}'') \leq \#(\text{SEQ}') \).

Using (1), \( \#(\text{SEQ}'') < \#(\text{SEQ}) \). So, by the induction hypothesis, there exists an essential subsequence \( \text{SEQ}''' \) of \( \text{SEQ}'' \), which is also a subsequence of \( \text{SEQ} \), on \( \Sigma \rightarrow A \triangleright B \). Hence, we obtain the lemma. \( \triangleright \)

**Lemma 3.7.** Let \( \text{SEQ} \) be a sequence on \( \Sigma \rightarrow A \triangleright B \) and let it be that \( \Sigma' \subseteq \Sigma \). If there exists a mapping \( f \) on \( \text{SEQ}/\Sigma' \), then there exists an essential subsequence \( \text{SEQ}' \) of \( \text{SEQ} \) on \( \Sigma' \rightarrow A \triangleright B \).

Proof. By Lemma 3.5, there exist a subsequence \( \text{SEQ}' \) of \( \text{SEQ} \) on \( \Sigma \rightarrow A \triangleright B \) and a pseudo-onto mapping on \( \text{SEQ}'/\Sigma' \rightarrow A \triangleright B \). Using Lemma 3.3, \( \text{SEQ}' \) is also on \( \Sigma' \rightarrow A \triangleright B \). Using Lemma 3.6, there exists an essential subsequence \( \text{SEQ}'' \) of \( \text{SEQ}' \) on \( \Sigma' \rightarrow A \triangleright B \).

**Lemma 3.8.** If there exists a cut-free proof figure for \( A, \Delta \triangleright \bot \rightarrow \Delta \), then there exists a cut-free proof figure for \( \Delta \triangleright \bot \rightarrow A \triangleright \bot \) in GIK4 such that \( A \triangleright \bot \) does not occur in any succedent of upper sequents of the inference rule introducing the end sequent.

Proof. It is easily seen that the sequence
\[
A \rightarrow \Delta, \bot \rightarrow \bot
\]
is essential on \( \Delta \triangleright \bot \rightarrow A \triangleright \bot \). So, the following figure convinces us that we can construct a cut-free proof figure for \( \Delta \triangleright \bot \rightarrow A \triangleright \bot \) satisfying the condition using a cut-free proof figure for \( A, \Delta \triangleright \bot \rightarrow \Delta \).

\[
\Delta \triangleright \bot \rightarrow A \triangleright \bot
\]

**Definition 3.9.** The degree \( d(A) \) of a formula \( A \) is redefined inductively as follows:
(1) \( d(p) = 1 \),
(2) \( d(\bot) = 0 \),
(3) \( d(A \land B) = d(A \lor B) = d(A \supset B) = d(A \triangleright B) = d(A) + d(B) + 1 \).

**Proof of Lemma 3.2.** The degree \( d(P) \), the left rank \( R_l(P) \) and the right rank \( R_r(P) \) of \( P \) are defined as usual. We use an induction on \( R_l(P) + R_r(P) + \omega d(P) \). We only treat the case that \( P \) is of the form:
\[
\vdots
\frac{(\text{SEQ})^*}{\Sigma^l \rightarrow C \triangleright D}
\frac{(\text{SEQ})^*}{\Sigma^r \rightarrow A \triangleright B}
\frac{C \triangleright D, \Sigma^r \rightarrow A \triangleright B}{\Sigma^r, \Sigma^l \rightarrow A \triangleright B}
\]
where \( C \triangleright D \notin \Sigma^l \) and two sequences \( \text{SEQ}^l \) and \( \text{SEQ}^r \) are essential on
\[ \Sigma^f \rightarrow C \succ D \text{ and } C \succ D, \Sigma^r \rightarrow A \succ B, \]
respectively. Without loss of generality, we assume that \( P \) is of the following form if \( C = \bot \).

\[
\begin{array}{c}
\bot \rightarrow \\
\bot \rightarrow D \\
\bot, D \succ \bot \rightarrow D \\
\Sigma^f \rightarrow \bot \succ D \\
\Sigma^f, \Sigma^r_{\bot \succ D} \rightarrow A \succ B
\end{array}
\]

Since \( \text{SEQ}^f \) and \( \text{SEQ}^r \) are essential, there exist pseudo-onto mappings \( f^f \) and \( f^r \) on \( \text{SEQ}^f / \Sigma^f \rightarrow C \succ D \) and \( \text{SEQ}^r / C \succ D, \Sigma^r \rightarrow A \succ B \), respectively. Using \( f^r \), we define a sequence \( \text{SEQ} \) as follows.

Let \( o'_1, \ldots, o'_n \) be an enumeration of the occurrences of sequents in \( \text{SEQ}^r \) such that

\[
C \in \text{Su}(o'_1)_B \text{ and } \text{an}(f^r(o'_1, C)) = D.
\]

So, \( \text{an}(f^r(o'_1, C)) = \cdots = \text{an}(f^r(o'_n, C)) = D \). Since \( \text{SEQ}^r \) is essential (i.e., by Definition 1.7(3)), we have \( f^r(o'_1, C) = \cdots = f^r(o'_n, C) \). We put

\[
o'_d = f^r(o'_1, C) = \cdots = f^r(o'_n, C).
\]

Here we note that for any occurrence \( o \) in \( \text{SEQ} \) and for any \( E \in \text{Su}(o)_B \),

\[
E \succ \text{an}(f^r(o, E)) = C \succ D \text{ implies } o \in \{o'_1, \ldots, o'_n\}, \ldots (\ast 1)
\]

Let \( o'_0 \) be the occurrence of the sequent at the top of \( \text{SEQ}^r \) and let \( S'_0 \) be the sequent at \( o'_0 \). For a sequent \( S^f \) in \( \text{SEQ}^f \) and \( i \in \{1, \ldots, n\} \), we define \( g_i(S^f) \) as follows.

\[
g_i(S^f) = \begin{cases} 
\text{an}(o'_i) \rightarrow \text{Su}(o'_i)_{C, D}, \text{Su}(S^f)_D & \text{if } S = S'_0 \\
\text{an}(S^f) \rightarrow \text{Su}(o'_i)_C, \text{Su}(S^f)_D & \text{otherwise}
\end{cases}
\]

Let \( g_i(\text{SEQ}^f) \) be the sequence obtained from \( \text{SEQ}^f \) by replacing each sequent \( S^f \) by \( g_i(S^f) \), respectively. For the occurrence \( o'_i \) of \( S^f \) in \( \text{SEQ}^f \), \( g_i(o'_i) \) denotes the occurrence of \( g_i(S^f) \) in \( g_i(\text{SEQ}^f) \). The sequence \( \text{SEQ} \) denotes the sequence obtained from \( \text{SEQ}^f \) by replacing each occurrence \( o'_i \) by the sequence \( g_i(\text{SEQ}^f) \), respectively.

Now, we show the following:

(3) \( \text{SEQ} \) is on \( C \succ D, \Sigma^f, \Sigma^r \rightarrow A \succ B \),

(4) there exists a mapping \( f \) on \( \text{SEQ} / \Sigma^f \cup \Sigma^r_{C \succ D} \),

(5) there exists a cut-free proof figure for each sequent in \( \text{SEQ}^r \).

By (3), (4) and Lemma 3.7, we can obtain an essential subsequence of \( \text{SEQ} \) on the end sequent of \( P \) and using (5), we can obtain the lemma.

For (3): It is sufficient to show the following two:

(3.1) each sequent \( S \) in \( \text{SEQ} \) is on \( C \succ D, \Sigma^f, \Sigma^r \rightarrow A \succ B \),

(3.2) \( \text{an}(o) = A \), for some occurrence \( o \) in \( \text{SEQ} \).

To show (3.1), we divide into the following three cases.

(i) The case that \( S \) does not occur in any \( g_i(\text{SEQ}^f) \): Then \( S \) occurs in \( \text{SEQ}^r \), and so, on \( C \succ D, \Sigma^r \rightarrow A \succ B \). Using Remark 1.4(1), we obtain (3.2).

(ii) The case that \( S \) occurs at \( g_i(o'_i) \) for some \( i \): Then \( S = g_i(S'_0) \), and it is of the form

\[
\text{an}(o'_i) \rightarrow \text{Su}(o'_i)_C, \text{Su}(S'_0)_D, \text{Su}(o'_i)_C.
\]

Since sequents occurring at \( o'_i \) and \( o'_d \) are on \( C \succ D, \Sigma^r \rightarrow A \succ B \), we have

\[
\text{an}(o'_i) \in \{A\} \cup \{E \mid E \succ F \in \{C \succ D\} \cup \Sigma^r\},
\]

\[
\text{Su}(o'_d) \subseteq \{E \mid E \succ F \in \{C \succ D\} \cup \Sigma^r\} \cup \{B\},
\]

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Similarly, since \( S_0^0 \) is on \( \Sigma^f \to C \supset D \), we have

\[
\text{Su}(S_0^0) \subseteq \{ E \mid E \supset F \in \{ C \supset D \} \cup \{ B \} \}.
\]

and so,

\[
\text{Su}(S_0^0)_D \subseteq \{ E \mid E \supset F \in \Sigma^f \}.
\]

Hence

\[
\text{an}(S) = \text{an}(o^*_i) \in \{ A \} \cup \{ F \mid E \supset F \in \{ C \supset D \} \cup \Sigma^f \cup \Sigma^r \}
\]

and

\[
\text{Su}(S) = \text{Su}(o^*_i) \cup \text{Su}(S_0^0)_D \cup \text{Su}(o^*_i)_C \subseteq \{ E \mid E \supset F \in \{ C \supset D \} \cup \Sigma^f \cup \Sigma^r \} \cup \{ B \}.
\]

Hence we obtain (3.1).

(iii) The case that \( S \) does not occur at \( g_i(o^*_i) \) for any \( i \), but occurs in \( g_i(\text{SEQ}^f) \) for some \( i \): In a similar way to the case above, we obtain (3.1).

We show (3.2). Since \( \text{SEQ}^f \) is essential, \( \text{an}(o^r) = A \) for some occurrence \( o^r \) of a sequent in \( \text{SEQ}^f \). If \( o^r \notin \{ o^*_1, \ldots, o^*_n \} \), then \( o^r \) also occurs in \( \text{SEQ} \), and \( \text{an}(o^r) = A \). If \( o^r = o^*_i \) for some \( i \), then \( A = \text{an}(o^r) = \text{an}(o^*_i) = \text{an}(g_i(o^*_i)) \), and \( g_i(o^*_i) \) occurs in \( \text{SEQ} \).

For (4): We define a mapping \( f \) on \( \text{SEQ}^f \cap \Sigma^f \cup \Sigma^r \cup \Sigma^r \). Let \( o \) be an occurrence in \( \text{SEQ} \) and let \( E \) be a formula in \( \text{Su}(o)_B \). We divide into the same three cases as the ones we used to show (3.1). In each case, we define \( f(o, E) \) in \( \text{SEQ} \) and show the following two:

\[
\begin{align*}
(4.1) & \quad E \supset \text{an}(f(o, E)) \in \Sigma^f \cup \Sigma^r, \\
(4.2) & \quad d_{\text{SEQ}}(o) > d_{\text{SEQ}}(f(o, E)).
\end{align*}
\]

From these two conditions, we will see that \( f \) is a mapping on \( \text{SEQ}^f \cap \Sigma^f \cup \Sigma^r \).

(i) The case that \( o \) does not occur in any \( g_i(\text{SEQ}^f) \): We note that \( o \) occurs in \( \text{SEQ}^f \), and

\[
o \notin \{ o^*_1, \ldots, o^*_n \} \quad (*2)
\]

Now, we define \( f(o, E) \) as follows.

\[
f(o, E) = \begin{cases} 
   f^r(o, E) & \text{if } f^r(o, E) \notin \{ o^*_1, \ldots, o^*_n \} \\
   g_i(o^*_i) & \text{if } f^r(o, E) = o^*_i \text{ for some } i
\end{cases}
\]

We show (4.1). If \( f^r(o, E) = o^*_i \) for some \( i \), then

\[
\text{an}(f(o, E)) = \text{an}(g_i(o^*_i)) = \text{an}(o^*_i) = \text{an}(f^r(o, E));
\]

if not, we also have

\[
\text{an}(f(o, E)) = \text{an}(f^r(o, E)).
\]

Since \( f^r \) is on \( \text{SEQ}^f \cap \{ C \supset D \} \cup \Sigma^r \),

\[
E \supset \text{an}(f(o, E)) = E \supset \text{an}(f^r(o, E)) \in \{ C \supset D \} \cup \Sigma^r.
\]

On the other hand, by (*1) and (*2), we have

\[
E \supset \text{an}(f^r(o, E)) \neq C \supset D.
\]

Hence

\[
E \supset \text{an}(f(o, E)) = E \supset \text{an}(f^r(o, E)) \in \Sigma^r_{C \supset D}.
\]

We show (4.2). Since \( f^r \) is on \( \text{SEQ}^f \cap \{ C \supset D \} \cup \Sigma^r \),

\[
d_{\text{SEQ}^f}(o) > d_{\text{SEQ}^f}(f^r(o, E)).
\]
If \( f^*(o, E) \notin \{o_1^*, \ldots, o_n^*\} \), then \( f^*(o, E) \) occurs in \( \text{SEQ} \), and so,
\[
d_{\text{SEQ}}(o) > d_{\text{SEQ}}(f^*(o, E)) = d_{\text{SEQ}}(f(o, E)).
\]
If \( f^*(o, E) = o_i^* \) for some \( i \), then
\[
d_{\text{SEQ}^*}(o) > d_{\text{SEQ}^*}(f^*(o, E)) = d_{\text{SEQ}^*}(o_i^*),
\]
and so,
\[
d_{\text{SEQ}}(o) > d_{\text{SEQ}}(g_i(o_0^i)) = d_{\text{SEQ}}(f(o, E)).
\]

(ii) The case that \( o \) occurs at \( g_i(o_0^i) \) for some \( i \): Then
\[
\text{Su}(o)_B = \text{Su}(g_i(o_0^i))_B = (\text{Su}(o_i^*)_C \cup \text{Su}(o_0^i)_D \cup \text{Su}(o_0^i)_D)_B.
\]
Note that \( f^*(o_d^*, E) \) occurs in \( \text{SEQ} \) if \( E \in \text{Su}(o_d^*_B) \), and \( g_i(f^*(o^*, E)) \) occurs in \( \text{SEQ} \) if \( E \in \text{Su}(o^*_D) \).

Now, \( f(o, E) \) is defined as follows.
\[
f(o, E) = \begin{cases} 
  f^*(o_i^*, E) & \text{if } E \in (\text{Su}(o_i^*)_C)_B \text{ and } f^*(o_i^*, E) \notin \{o_1^*, \ldots, o_n^*\} \\
  g_j(o_0^i)_B & \text{if } E \in (\text{Su}(o_i^*)_C)_B \text{ and } f^*(o_i^*, E) = o_j^* \text{ for some } j \\
  f^*(o_d^*_B, E) & \text{if } E \in \text{Su}(o_d^*_B) \\
  g_i(f^*(o^*, E)) & \text{if } E \in (\text{Su}(o^*_D)_D)_B 
\end{cases}
\]

We show (4.1).
If \( E \in (\text{Su}(o_i^*)_C)_B \) and \( f^*(o_i^*, E) \notin \{o_1^*, \ldots, o_n^*\} \), then
\[
E \triangleright an(f(o, E)) = E \triangleright an(f^*(o_i^*, E)) \in \{C \supset D\} \cup \Sigma^r.
\]
Since \( E \in (\text{Su}(o_i^*)_C)_B \), we have \( E \neq C \), and so, \( E \triangleright an(f(o, E)) \in \Sigma^r_{C \supset D} \).
If \( E \in (\text{Su}(o_i^*)_C)_B \) and \( f^*(o_i^*, E) = o_j^* \), then
\[
E \triangleright an(f(o, E)) = E \triangleright an(g_j(o_0^i)) = E \triangleright an(o_j^*) = E \triangleright an(f^*(o_i^*, E)) \in \{C \supset D\} \cup \Sigma^r.
\]
Since \( E \in (\text{Su}(o_i^*)_C)_B \), we have \( E \triangleright an(f(o, E)) \in \Sigma^r_{C \supset D} \), similarly.
If \( E \in \text{Su}(o_d^*_B)_D \), then
\[
E \triangleright an(f(o, E)) = E \triangleright an(f^*(o_d^*_B, E)) \in \{C \supset D\} \cup \Sigma^r.
\]
By (1) and \( o_d^* \notin \{o_1^*, \ldots, o_n^*\} \), we have \( E \triangleright an(f^*(o_d^*_B, E)) \neq C \supset D \). Hence, \( E \triangleright an(f(o, E)) \in \Sigma^r_{C \supset D} \).
If \( E \in (\text{Su}(o_d^*_B)_D)_B \), then
\[
E \triangleright an(f(o, E)) = E \triangleright an(g_i(f^*(o^*_B, E))) = E \triangleright an(f^*(o^*, E)) \in \Sigma^r.
\]
We show (4.2).
If \( E \in (\text{Su}(o_i^*)_C)_B \) and \( f^*(o_i^*, E) \notin \{o_1^*, \ldots, o_n^*\} \), then
\[
d_{\text{SEQ}^*}(o_i^*) > d_{\text{SEQ}^*}(f^*(o_i^*, E)),
\]
and so,
\[
d_{\text{SEQ}}(o) > d_{\text{SEQ}}(f^*(o_i^*, E)) = d_{\text{SEQ}}(f(o, E)).
\]
If \( E \in (\text{Su}(o_i^*)_C)_B \) and \( f^*(o_i^*, E) = o_j^* \), then
\[
d_{\text{SEQ}^*}(o_i^*) > d_{\text{SEQ}^*}(o_j^*),
\]
and so,
\[
d_{\text{SEQ}}(o) > d_{\text{SEQ}}(g_i(o_0^i)) = d_{\text{SEQ}}(f(o, E)).
\]
If $E \in \text{Su}(o_d^i)_B$, then
\[ d_{\text{SEQ}^i}(o_d^i) > d_{\text{SEQ}^i}(o_d^i) > d_{\text{SEQ}^i}(f(o_d^i, E)), \]
and so,
\[ d_{\text{SEQ}^i}(o) > d_{\text{SEQ}^i}(f(o_d^i, E)) = d_{\text{SEQ}^i}(f(o, E)). \]
If $E \in (\text{Su}(o^i_D)_D)_B$, then
\[ d_{\text{SEQ}^i}(o^i) > d_{\text{SEQ}^i}(f(o^i, E)), \]
and so,
\[ d_{\text{SEQ}^i}(o) = d_{\text{SEQ}^i}(g(o^i)) > d_{\text{SEQ}^i}(g_i(f(o^i, E))) = d_{\text{SEQ}^i}(f(o, E)). \]

(iii) The case that $S$ does not occur at $g_i(o^i_0)$ for any $i$, but occurs in $g_i(\text{SEQ}^i)$ for some $i$: By the definition of $g_i(\text{SEQ}^i)$, there exists an occurrence $o^i$, which is different from $o^i_0$, in $\text{SEQ}^i$ such that $o = g_i(o^i)$. Also
\[ \text{Su}(o)_B = \text{Su}(g_i(o^i))_B = (\text{Su}(o_d^i) \cup \text{Su}(o^i_D))_B. \]
$f(o, E)$ is defined as follows:
\[
\begin{align*}
f(o, E) &= \begin{cases} 
  f^*(o_d^i, E) & \text{if } E \in \text{Su}(o_d^i)_B \\
  g_i(f^*(o^i, E)) & \text{if } E \in (\text{Su}(o^i_D)_D)_B
\end{cases} 
\end{align*}
\]
(4.1) and (4.2) can be shown in a way similar to the case (ii) above.

For (5): Let $S^r_i$ and $S^r_d$ be the sequents occurring at $o^r_d$ and $o^r_d$, respectively. Then $(S^r_i)^*$ and $(S^r_d)^*$ occurring in $\text{SEQ}^r$ are of the following forms:
\[
\begin{align*}
(S^r_i)^* : \text{an}(S^r_i), & \text{Su}(S^r_i) \triangleright \bot \rightarrow \text{Su}(S^r_i), \\
(S^r_d)^* : D, & \text{Su}(S^r_d) \triangleright \bot \rightarrow \text{Su}(S^r_d),
\end{align*}
\]
where $C \in \text{Su}(S^r_i)$ and $B \in \text{Su}(S^r_i) \cup \text{Su}(S^r_d)$. Also $(S^r_0)^*$, which is the sequent occurring at the top of $(\text{SEQ}^r)^*$, is of the form
\[
(S^r_0)^* : C, \text{Su}(S^r_0) \triangleright \bot \rightarrow \text{Su}(S^r_0),
\]
where $D \in \text{Su}(S^r_0)$. Hence there exist cut-free proof figures $P^r_i$, $P^r_d$ and $P^r_0$ for the sequents obtained from $(S^r_i)^*$, $(S^r_d)^*$ and $(S^r_0)^*$, respectively.

Using Lemma 3.8, there exist cut-free proof figures for
\[ \text{Su}(S^r_i) \triangleright \bot \rightarrow D \triangleright \bot \]
and
\[ \text{Su}(S^r_0) \triangleright \bot \rightarrow C \triangleright \bot. \]
Let $Q^r_d$ and $Q^r_0$ be the proof figures for the sequents above obtained by the way in Lemma 3.8.

Let $T$ be a sequent in $\text{SEQ}^r$. Then there exists a sequent $S$ in $\text{SEQ}$ such that $S^r = T$. We divide into the same three cases as the ones we used to show (3.1) and (4).

(i) The case that $S$ does not occur in any $g_i(\text{SEQ}^i)$: Then $S$ occurs in $\text{SEQ}^r$, and so, there exists a cut-free proof figure for $S^r = T$.

(ii) The case that $S$ occurs at $g_i(o^i_0)$ for some $i$: $S^r$ is of the form
\[
\text{an}(S^r_i), \text{Su}(S^r_i) \triangleright \bot, \text{Su}(S^r_d) \triangleright \bot, \text{Su}(S^r_0)_D \triangleright \bot \rightarrow \text{Su}(S^r_i)_C, \text{Su}(S^r_d), \text{Su}(S^r_0)_D.
\]
We divide into the following two subcases.

(ii.a) The subcase that $C \neq \bot$: By $P^r_i$, $Q^r_0$ and cut, we have the following proof figure $P_1$:
\[
\frac{Q^r_0}{\text{Su}(S^r_0) \triangleright \bot, (\{\text{an}(S^r_i)\} \cup \text{Su}(S^r_i) \triangleright \bot)_C \triangleright \bot \rightarrow \text{Su}(S^r_i)}
\]


We note that \( d(P_1) = d(C) \leq d(C \triangleright D) = d(P) \) and \( R_r(P_1) < R_r(P) \). By Lemma 3.8, \( R_l(P_1) = 1 = R_l(P) \). So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of \( P_1 \).

Using \( T \rightarrow \), possibly several times, we obtain a cut-free proof figure \( P_2 \) for
\[
\text{an}(S'_i), \text{Su}(S'_0) \triangleright 0, \text{Su}(S'_r) \triangleright 0 \rightarrow \text{Su}(S'_r).
\]

Using \( P_0^f \) and cut, we have the following proof figure \( P_3 \):
\[
\begin{array}{c}
\text{an}(S'_r), \text{Su}(S'_r) \triangleright 0, \text{Su}(S'_r) \triangleright 0 \rightarrow \text{Su}(S'_r)
\end{array}
\]

We note that \( d(P_3) = d(C) < d(C \triangleright D) = d(P) \). So, by the induction hypothesis, we obtain a cut-free proof figure \( P_4 \) for the end sequent of \( P_3 \). Using \( Q_d^r \) and cut, we have the following proof figure \( P_5 \):
\[
\begin{array}{c}
\text{Su}(S'_r) \triangleright 0, \text{Su}(S'_0) \triangleright 0 \rightarrow \text{Su}(S'_r) \cap \text{Su}(S'_0)
\end{array}
\]

Since \( C \neq 0 \), we have \( d(C) > 0 \), and so, \( d(P_5) = d(D) + 1 = d(D) + 1 < d(C) + d(D) + 1 = d(C) + d(D) = d(P) \). So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of \( P_5 \). Using \( T \rightarrow \), possibly several times, we obtain a cut-free proof figure for \( S^* = T \).

(ii.b) The case that \( C = 0 \): By \( P_6^r \) and cut, we have the following proof figure \( P_7 \):
\[
\begin{array}{c}
\text{an}(S'_r), \text{Su}(S'_r) \triangleright 0, \text{Su}(S'_0) \triangleright 0 \rightarrow \text{Su}(S'_r) \cap \text{Su}(S'_0)
\end{array}
\]

We note that \( d(P_7) = d(D) < d(C \triangleright D) = d(P) \). So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of \( P_7 \). Using \( T \rightarrow \), possibly several times, we obtain a cut-free proof figure for \( S^* = T \).

Using the axiom \( \perp \rightarrow \) and cut, we have the following cut-free proof figure \( P_9 \):
\[
\begin{array}{c}
\text{an}(S'_r), \text{Su}(S'_r) \triangleright 0 \rightarrow \text{Su}(S'_r)
\end{array}
\]

We note that \( d(P_9) = d(\perp) < d(\perp \triangleright D) = d(P) \). So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of \( P_9 \). Using \( T \rightarrow \) and \((\rightarrow T)\), possibly several times, we obtain a cut-free proof figure for \( S^* = T \).

(iii) The case that \( S \) does not occur at \( g_i(\alpha'_0) \) for any \( i \), but occurs in \( g_i(\text{SEQ}^f) \) for some \( i \): There exists a sequent \( S' \) in \( \text{SEQ}^f \) such that \( g_i(S') = S \) and \( S \neq S'_0 \). On the other hand, \( S^* \) is of the form
\[
\text{an}(S'), \text{Su}(S'_0) \triangleright 0, \text{Su}(S') \triangleright 0 \rightarrow \text{Su}(S'_0) \cap \text{Su}(S')
\]
where \( B \in \text{Su}(S^t_d) \). Since \( S^t \) occurs in \( \text{SEQ}'^t \), \((S^t)^* \) occurs in \( (\text{SEQ}'^t)^* \), and so, there exists a cut-free proof figure \( P^t \) for

\[
S^t : \text{an}(S^t), \text{Su}(S^t) \triangleright \bot \rightarrow \text{Su}(S^t).
\]

Using \( Q^t_d \) and cuts, we have the following cut-free proof figure \( P_{11} \):

\[
\frac{Q^t_d}{\text{Su}(S^t_d) \triangleright \bot, \{\text{an}(S^t)\} \cup \text{Su}(S^t) \triangleright \bot}_{D, \bot} \rightarrow \text{Su}(S^t)
\]

On the other hand, from our first assumption, \( \text{SEQ}' \) consists of only one sequent if \( C = \bot \). Here, however, \( \text{SEQ}' \) has at least two sequents \( S^t_0 \) and \( S^t \). Hence we have \( C \neq \bot \) (i.e., \( d(C) > 0 \)), and so, \( d(P_{11}) = d(D \triangleright \bot) = d(D) + 1 < d(C) + d(D) + 1 = d(C \triangleright D) = d(P) \). So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of \( P_{11} \). Using \( P^t_d \) and cuts, we have the following cut-free proof figure \( P_{12} \):

\[
\frac{P_{11}}{\text{Su}(S^t_d) \triangleright \bot, \{\text{an}(S^t)\} \cup \text{Su}(S^t) \triangleright \bot}_{D, \bot} \rightarrow \text{Su}(S^t_d), \text{Su}(S^t_d)}
\]

We note \( d(P_{12}) = d(D) < d(C \triangleright D) = d(P) \). So, by the induction hypothesis, we obtain a cut-free proof figure for the end sequent of \( P_{12} \). Using \( (T \rightarrow) \), possibly several times, we obtain a cut-free proof figure for \( S^* = T \).

By (3),(4) and Lemma 3.7, there exists an essential subsequence \( \text{SEQ}' \) of \( \text{SEQ} \) on \( \Sigma^t, \Sigma^r_{C \triangleright D} \rightarrow A \triangleright B \). So,

\[
(\text{SEQ}'^t)^* \rightarrow \Sigma^t, \Sigma^r_{C \triangleright D} \rightarrow A \triangleright B
\]

is an inference rule. Since \( \text{SEQ}' \) is a subsequence of \( \text{SEQ} \), \( (\text{SEQ}'^t)^* \) is of \( \text{SEQ}^* \), and so, every sequent in \( (\text{SEQ}'^t)^* \) also occurs in \( \text{SEQ}^* \). Hence there exist a cut-free proof figure for each sequent in \( (\text{SEQ}'^t)^* \). Using the inference rule above, we obtain a cut-free proof figure for

\[
\Sigma^t, \Sigma^r_{C \triangleright D} \rightarrow A \triangleright B,
\]

the end sequent of \( P \).

**Definition 3.10.** The set \( \text{Sub}^>(A) \) is defined inductively as follows:

1. \( \text{Sub}^>(B) = \text{Sub}(B) \), for a non-modal formula \( B \),
2. \( \text{Sub}^>(C \triangleright D) = \text{Sub}^>(C) \cup \text{Sub}^>(D) \cup \{\bot, C \triangleright \bot, D \triangleright \bot\} \).

We note that \( \text{Sub}^>(A) \) is finite.

**Corollary 3.11.** Let it be that \( \Gamma \rightarrow \Delta \in \text{GIK}4 \). Then there exists a cut-free proof figure \( P \) satisfying the following condition for any formula \( A \):

\[
\text{if } A \text{ occurs in } P, \text{ then } A \in \text{Sub}^>(B) \text{ for some } B \text{ occurring in } \Gamma \rightarrow \Delta.
\]

By Corollary 1.8 and Corollary 3.11, our system \( \text{GIK}4 \) gives a decision procedure for \( \text{IK}4 \).

**References**


