A Simplified Morass using Partially Frozen Finite Conditions

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Abstract

Jensen invented the morass around 1970, which he used to solve open problems in model theory. Jensen's morass appears to be strongly related to the fine structure of the constructible universe of Godel; however, Velleman provided new points of view to simplify Jensen's morass around 1980. He showed the existence of his simplified morass was equivalent to Jensen's. Velleman also solved problems in set theory and topology using his simplified approach.

We focus on one of Velleman's simplified morasses in the forcing construction, within which several partially ordered sets (posets) are known; some consist of countable conditions and others are finite. The finite conditions are usually formed together with finite fragments of fast functions. The fragments control the continuous growth of the forced simplified morass. This study presents a new poset of finite conditions that bear no finite fragments of fast functions. They are partially ordered so that the forced object reveals its unstable shape by gradually freezing its body. This new poset is proper and has the correct chain condition under the continuum hypothesis.

Introduction

Our studies ([M1], [M2]) propose partially ordered sets (posets) that comprise finite conditions to force a simplified $(\omega_1, 1)$-morass. As [M1] and [M2] incorporate fast functions, it is possible to acquire continuity on the part of the family $\mathcal{N}$ forced. Namely, the formulation of the item called (partition) in [M1] and [M2] that involved the notion of $\lim(\mathcal{N}) = \{N \in \mathcal{N} \mid \bigcup(\mathcal{N} \cap N) = N\}$. We also propose a poset that consists of two types of elementary substructures to force a simplified $(\omega_2, 1)$-morass in [M3]; however, in this forcing, it appears impossible to acquire continuity regardless of relevant cofinalities. This observation led to the formulation of the item (2-partition) in [M3]. This study presents a new poset to force a simplified $(\omega_1, 1)$-morass similar to [M3], which involves the countable elementary substructures with no fast functions. The morass and related matters can be found in [J] and [D]. The simplified morass and related matters can be found in [V] and [D].

Notation. Let us denote $H_{\omega_2} := \{x \mid$ the transitive closure of x with respect to the binary relation $\in$ is of a size less than $\omega_2\}$. Let $(H_{\omega_2}, \in, \cdots)$ be your favorite relational structure. Somewhat abusively, denote $C_0 := \{N \in [H_{\omega_2}]^{\omega} \mid$ the naturally induced substructure $(N, \in, \cdots)$

of $(H_{\omega_2}, \in, \cdots)$ by N is an elementary one$\}$. For $N \in C_0$, we know $\omega_1 \cap N < \omega_1$ and denote $\alpha_N := \omega_1 \cap N$. For $N, N' \in C_0$, write $N =_{\omega_1} N'$ if $\alpha_N = \alpha_{N'}$ and $N <_{\omega_1} N'$, if $\alpha_N < \alpha_{N'}$. For any map $\phi: N \longrightarrow N'$ and $A \subseteq N$, denote $\phi[A] := \{\phi(x) \mid x \in A\}$.

We intend to exhibit a proper poset P that has the ω_2-cc under the continuum hypothesis (CH) and forces $\mathcal{N}$ over the ground model V s.t.

- (el) $\mathcal{N} \subseteq (C_0)^V$.
- (iso) If $N, N' \in \mathcal{N}$ with $N =_{\omega_1} N'$, then there exists the unique isomorphism $\phi_{NN'}: (N, \in, \cdots) \longrightarrow (N', \in, \cdots)$ s.t. $\phi_{NN'}(x) = x$ for all $x \in N \cap N'$.
- (up) If $N_3, N_2 \in \mathcal{N}$ with $N_3 <_{\omega_1} N_2$, then there exists $N_1 \in \mathcal{N}$ s.t. $N_3 \in N_1 =_{\omega_1} N_2$.
- (down) If $N_1, N_2, N_3 \in \mathcal{N}$ with $N_3 \in N_1 =_{\omega_1} N_2$, then $\phi_{N_1N_2}(N_3) \in \mathcal{N}$.
- (2-partition) $\mathcal{N} = \operatorname{suc}(\mathcal{N}) \cup \operatorname{dir}(\mathcal{N})$, where for $N \in \mathcal{N}$,
  $N \in \operatorname{suc}(\mathcal{N})$, if there exists (N_1, N_2) s.t.
  $*\ \mathcal{N} \cap \mathrm{N} = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$.
  $*$ If $N_1 \neq N_2$, then $N_1 =_{\omega_1} N_2$ and $\Delta := (\omega_2 \cap N_1) \cap (\omega_2 \cap N_2) < (\omega_2 \cap N_1) \setminus \Delta < (\omega_2 \cap N_2) \setminus \Delta \neq \emptyset$.
  $N \in \operatorname{dir}(\mathcal{N})$, if $(\mathcal{N} \cap N, \in)$ is directed, i.e., for any $X, Y \in \mathcal{N} \cap N$, there exists $Z \in \mathcal{N} \cap N$ s.t. $X, Y \in Z$.
- (cof) $\bigcup \mathcal{N} = (\mathrm{H}_{\omega_2})^V$.

Note. If $\mathcal{N} \cap N = \emptyset$, then $N \in \operatorname{dir}(\mathcal{N})$ by the definition. Even if $N \in \operatorname{dir}(\mathcal{N})$, it is not intended to have $N = \bigcup(\mathcal{N} \cap N)$. This is the new point of view in the study resulting in the new poset P. In [M3], we proposed a similar representation of a simplified $(\omega_2, 1)$-morass.

The Projection of $\mathcal{N}$

Let $\mathcal{N}$ be as in the previous section. Hence, $\mathcal{N}$ satisfies (el), (iso), (up), (down), (2-partition), and (cof) in the generic extension where the cofinalities and so the cardinalities are all preserved. We consider the projection

$$\mathcal{A} = \mathcal{N} \lceil \omega_2 := \{\omega_2 \cap N \mid N \in \mathcal{N}\}.$$

Lemma. $\mathcal{A} = \mathcal{N} \lceil \omega_2$ is a (non-neat) simplified $(\omega_1, 1)$-morass in the generic extension.

Proof. We check the following 6 items from Definition 2.6 on p.259 in [V].

- (well founded) There exists no proper $\subseteq$-descending infinite sequence of members of $\mathcal{A}$. Let $\operatorname{rank}(a)$ denote the rank of $a \in \mathcal{A}$ w.r.t. the well founded relation.
- (homogeneous) For any $a, b \in \mathcal{A}$ with $\operatorname{rank}(a) = \operatorname{rank}(b)$, there exists the isomorphism $\phi_{ab}: (a, <) \longrightarrow (b, <)$ s.t. $\mathcal{A} \lceil b := \{y \in \mathcal{A} \mid y \subseteq b \text{ (proper)}\} = \{\text{the set of images } \phi_{ab}[x] \mid x \in \mathcal{A} \lceil a\}$.
- (locally small) For any $a \in \mathcal{A}$, let $|\mathcal{A} \lceil a|$ denote the size of $\mathcal{A} \lceil a$, then $|\mathcal{A} \lceil a| < \omega_1$.
- (directed) For any $a, b \in \mathcal{A}$, there exists $c \in \mathcal{A}$ with $a, b \subseteq c$.
- (locally almost directed) For any $a \in \mathcal{A}$, either $(\mathcal{A} \lceil a, \subseteq)$ is directed or there exists (a_1, a_2) s.t. $\operatorname{rank}(a_1) = \operatorname{rank}(a_2)$, $\Delta := a_1 \cap a_2 < a_1 \setminus \Delta < a_2 \setminus \Delta \neq \emptyset$, and $\mathcal{A} \lceil a = \{a_1, a_2\} \cup (\mathcal{A} \lceil a_1) \cup (\mathcal{A} \lceil a_2)$.

• (cover) $\bigcup \mathcal{A} = \omega_2$.

(well founded) The following suffices.

Claim.

(1) For any N, $X \in \mathcal{N}$ with $N =_{\omega_1} X$, the map ϕ_{NX} restricted to $\omega_2 \cap N$ is the isomorphism from $(\omega_2 \cap N, <)$ onto $(\omega_2 \cap X, <)$. In particular, the set of images $\phi_{NX}[\omega_2 \cap N] = \omega_2 \cap X$.
(2) For any $N, X \in \mathcal{N}$ with $\omega_2 \cap X \subseteq \omega_2 \cap N$ (proper), we have $X <_{\omega_1} N$.

Proof. (1) The set of ordinals $\omega_2 \cap N$ is a definable class with no parameters in N. For $x \in N$,

$$x \in \omega_2 \cap N \text{ iff } N \models \text{"Transitive}(x) \text{ and for all } y \in x \text{ Transitive}(y)\text{"}.$$

Hence, $\phi_{NX}[\omega_2 \cap N] = \omega_2 \cap X$.

(2) Take the intersections, we have $\omega_1 \cap (\omega_2 \cap X) \subseteq \omega_1 \cap (\omega_2 \cap N)$ and so $X \leq_{\omega_1} N$. Suppose on the contrary that $X =_{\omega_1} N$, then $\phi_{XN}[\omega_2 \cap X] = \omega_2 \cap N$. But $\omega_2 \cap X \subseteq X \cap N$. Hence, $\phi_{XN}[\omega_2 \cap X] = \omega_2 \cap X$ and so $\omega_2 \cap X = \omega_2 \cap N$. This is absurd.

(homogeneous) and (locally small) The following suffices.

Claim.

(1) For any $N \in \mathcal{N}$, let $a = \omega_2 \cap N$, then we have $\mathcal{A} \lceil a = \{\omega_2 \cap X \mid X \in \mathcal{N} \cap N\}$.
(2) For any N, $X \in \mathcal{N}$ with $N =_{\omega_1} X$, two structures $(\mathcal{A} \lceil (\omega_2 \cap N), \subseteq)$ and $(\mathcal{A} \lceil \omega_2 \cap X), \subseteq)$ are isomorphic by a restriction of ϕ_{NX}. In particular, $\text{rank}(\omega_2 \cap N) = \text{rank}(\omega_2 \cap X)$.
(3) For any $N, X \in \mathcal{N}$, $N =_{\omega_1} X$ iff $\text{rank}(\omega_2 \cap N) = \text{rank}(\omega_2 \cap X)$.

Proof. (1) For any $X \in \mathcal{N} \cap N$, we have $\omega_2 \cap X \in N$ and so $\omega_2 \cap X \subseteq \omega_2 \cap N$ (proper). It remains to show $\mathcal{A} \lceil a \subseteq \{\omega_2 \cap X \mid X \in \mathcal{N} \cap N\}$. Let $b \in \mathcal{A} \lceil a$. Then there exists $X \in \mathcal{N}$ with $\omega_2 \cap X = b$. Since $\omega_2 \cap X \subseteq \omega_2 \cap N$(proper), we have $X <_{\omega_1} N$. Take $Y \in \mathcal{N}$ s.t. $X \in Y =_{\omega_1} N$. Then $\phi_{YN}(X) \in \mathcal{N} \cap N$, $b = \omega_2 \cap X \subseteq X \cap N \subseteq Y \cap N$ and so $b = \phi_{YN}[b] = \phi_{YN}[\omega_2 \cap X] = \omega_2 \cap \phi_{YN}[X] = \omega_2 \cap \phi_{YN}(X)$.

(2) We have the unique isomorphism f from $(\omega_2 \cap N, \in)$ onto $(\omega_2 \cap X, \in)$. The restriction of ϕ_{NX} to $\omega_2 \cap N$ serves as the isomorphism f. Let $b \in \mathcal{A} \lceil (\omega_2 \cap N)$ with $b = \omega_2 \cap Y$ s.t. $Y \in \mathcal{N} \cap N$, then $f[b] = \phi_{NX}[b] = \omega_2 \cap \phi_{NX}(Y) \in \mathcal{A} \lceil (\omega_2 \cap X)$. Let b, $c \in \mathcal{A} \lceil (\omega_2 \cap N)$ with Y, $Z \in \mathcal{N} \cap N$ s.t. $b = \omega_2 \cap Y$ and $c = \omega_2 \cap Z$. Then $b \subseteq c$ (proper) iff $\omega_2 \cap Y \subseteq \omega_2 \cap Z$ (proper) iff $\omega_2 \cap \phi_{NX}(Y) \subseteq \omega_2 \cap \phi_{NX}(Z)$ (proper) iff $\phi_{NX}[b] \subseteq \phi_{NX}[c]$ (proper) iff $f[b] \subseteq f[c]$ (proper). For any $d \in \mathcal{A} \lceil (\omega_2 \cap X)$ with $d = \omega_2 \cap \phi_{NX}(Y)$ and $Y \in \mathcal{N} \cap N$, there exists $e = \omega_2 \cap Y \in \mathcal{A} \lceil (\omega_2 \cap N)$ with $f[e] = f[\omega_2 \cap Y] = \phi_{NX}[\omega_2 \cap Y] = \omega_2 \cap \phi_{NX}(Y) = d$.

(3) Suppose $N <_{\omega_1} X$. Then take $Y \in \mathcal{N}$ with $N \in Y =_{\omega_1} X$. Since $\omega_2 \cap N \in \mathcal{A} \lceil (\omega_2 \cap Y)$, we have $\text{rank}(\omega_2 \cap N) < \text{rank}(\omega_2 \cap Y) = \text{rank}(\omega_2 \cap X)$. Hence $\text{rank}(\omega_2 \cap N) < \text{rank}(\omega_2 \cap X)$. In particular, $\text{rank}(\omega_2 \cap N) = \text{rank}(\omega_2 \cap X)$ implies $N =_{\omega_1} X$.

(directed) Let a, $b \in \mathcal{A}$ with N, $X \in \mathcal{N}$ s.t. $a = \omega_2 \cap N$ and $b = \omega_2 \cap X$. Since $\bigcup \mathcal{N} = (H_{\omega_2})^V$, there exists $Z \in \mathcal{N}$ with $\{N, X\} \in Z$ and so $N, X \subseteq Z$. Then a, $b \subseteq \omega_2 \cap Z \in \mathcal{A}$.

(locally almost directed) Let $N \in \mathcal{N}$ and look at $\mathcal{A} \lceil (\omega_2 \cap N) = \{\omega_2 \cap X \mid X \in \mathcal{N} \cap N\}$.

Case. $N\in\mathrm{dir}(\mathcal{N})$: Then $(\mathcal{N}\cap N, \in)$ is directed and so $(\mathcal{A}\lceil(\omega_2\cap N), \subseteq)$ is directed. To see this, let $b, c\in\mathcal{A}\lceil(\omega_2\cap N)$ with $Y, Z\in\mathcal{N}\cap N$ s.t. $b=\omega_2\cap Y$ and $c=\omega_2\cap Z$. Take $W\in\mathcal{N}\cap N$ with $Y, Z\in W$ and so $Y, Z\subseteq W$. Then $b, c\subseteq\omega_2\cap W\in\mathcal{A}\lceil(\omega_2\cap N)$.

Case. $N\in\mathrm{suc}(\mathcal{N})$: Take (N_1, N_2) s.t. $\mathcal{N}\cap N=\{N_1, N_2\}\cup(\mathcal{N}\cap N_1)\cup(\mathcal{N}\cap N_2)$.

Subcase. $N_1=N_2$: Then $(\mathcal{A}\lceil(\omega_2\cap N), \subseteq)$ is directed, as for any $b\in\mathcal{A}\lceil(\omega_2\cap N)$, we have $b\subseteq\omega_2\cap N_1=\omega_2\cap N_2\in\mathcal{A}\lceil(\omega_2\cap N)$.

Subcase. $N_1\neq N_2$: Let $a_1=\omega_2\cap N_1$ and $a_2=\omega_2\cap N_2$. Then we have $\mathrm{rank}(a_1)=\mathrm{rank}(a_2)$, $\mathcal{A}\lceil(\omega_2\cap N)=\{a_1, a_2\}\cup(\mathcal{A}\lceil a_1)\cup(\mathcal{A}\lceil a_2)$, and $\Delta:=a_1\cap a_2<a_1\setminus\Delta<a_2\setminus\Delta\neq\emptyset$.

(cover) Since $\bigcup\mathcal{N}=(H_{\omega_2})^V$, we have $\bigcup\mathcal{A}=\omega_2$.

The New Poset P

The partial order with the item (freeze) on P appeared in [M3] that is somewhat new compared to the posets in [M1] and [M2]. In particular, no finite fragments of fast functions are incorporated in $p\in P$.

Definition. Let $p=\mathcal{N}_p\in P$, if

- (el with top) $\mathcal{N}_p$ is a finite subset of C_0 with an $N\in\mathcal{N}_p$ s.t. $\mathcal{N}_p=(\mathcal{N}_p\cap N)\cup\{N\}$.
- (iso) If $N, N'\in\mathcal{N}_p$ with $N=_{\omega_1}N'$, then there exists the unique isomorphism $\phi_{NN'}: (N, \in, \cdots)\longrightarrow(N', \in, \cdots)$ s.t. $\phi_{NN'}(x)=x$ for all $x\in N\cap N'$.
- (up) If $N_3, N_2\in\mathcal{N}_p$ with $N_3<_{\omega_1}N_2$, then there exists $N_1\in\mathcal{N}_p$ s.t. $N_3\in N_1=_{\omega_1}N_2$.
- (down) If $N_1, N_2, N_3\in\mathcal{N}_p$ with $N_3\in N_1=_{\omega_1}N_2$, then $\phi_{N_1N_2}(N_3)\in\mathcal{N}_p$.
- (pre-partition) For any $N\in\mathcal{N}_p$, either $\mathcal{N}_p\cap N=\emptyset$ or there exists (N_1, N_2) s.t.
  * $\mathcal{N}_p\cap \mathrm{N}=\{N_1, N_2\}\cup(\mathcal{N}_p\cap N_1)\cup(\mathcal{N}_p\cap N_2)$.
  * If $N_1\neq N_2$, then $N_1=_{\omega_1}N_2$ and $\Delta:=(\omega_2\cap N_1)\cap(\omega_2\cap N_2)<(\omega_2\cap N_1)\setminus\Delta<(\omega_2\cap N_2)\setminus\Delta\neq\emptyset$.

Prior to defining the partial order on P, we observe

Claim. (1) If N and N' served as in the item (el with top), then $N=N'$. We denote this unique element as N_p called the top element of p.

(2) (The immediate predecessors $\mathcal{X}_p^N$ of N in p) If $\mathcal{N}_p$ is known to just satisfy (el), (iso), (up), and (down), then for any $N\in\mathcal{N}_p$, there exists the unique $\mathcal{X}_p^N$ s.t. $X=_{\omega_1}Y$ for any $X, Y\in\mathcal{X}_p^N$ and $\mathcal{N}_p\cap N=\mathcal{X}_p^N\cup\ \bigcup\{\mathcal{N}_p\cap \mathrm{X}\mid \mathrm{X}\in\mathcal{X}_p^N\}$. Hence, if $\mathcal{N}_p\in P$ and $N\in\mathcal{N}_p$, then either $\mathcal{X}_p^N=\emptyset$ or $\mathcal{X}_p^N=\{N_1, N_2\}$ that is either a singleton set or consists of two elements with the specification.

Proof. (1) Let $(\mathcal{N}_p\cap N)\cup\{N\}=\mathcal{N}_p=(\mathcal{N}_p\cap N')\cup\{N'\}$. Suppose $N\neq N'$. Then $N\in N'$ and $N'\in N$. This is absurd.

(2) If $\mathcal{N}_p\cap N=\emptyset$. Then $\mathcal{X}_p^N=\emptyset$ is the only choice. Let us suppose $\mathcal{N}_p\cap N\neq\emptyset$. Let α^* be the largest member of $\{\alpha_Y\mid Y\in\mathcal{N}_p\cap N\}$. Let $\mathcal{X}=\{X\in\mathcal{N}_p\cap N\mid \alpha_X=\alpha^*\}$. Then $X=_{\omega_1}Y$ for any $X, Y\in\mathcal{X}$ and

$$\mathcal{N}_p \cap N = \mathcal{X} \cup \bigcup \{\mathcal{N}_p \cap X \mid X \in \mathcal{X}\}.$$

($\subseteq$): Let $Z \in \mathcal{N}_p \cap N$. If $\alpha_Z = \alpha^*$, then $Z \in \mathcal{X}$. If $\alpha_Z < \alpha^*$, then by (up) there exists $Y \in \mathcal{N}_p$ s.t. $Z \in Y$ and $\alpha_Y = \alpha^*$. By (up) again, there exists $N' \in \mathcal{N}_p$ s.t. $Y \in N' =_{\omega_1} N$. Notice $Z \in N \cap N'$. Then, by (iso) and (down), $Z = \phi_{N'N}(Z) \in \phi_{N'N}(Y) \in \mathcal{N}_p \cap N$. Since $\phi_{N'N}(Y) =_{\omega_1} Y$, we have $\phi_{N'N}(Y) \in \mathcal{X}$.

(uniqueness) Let $\mathcal{X}$ and $\mathcal{Y}$ served. Then

$$\mathcal{X} \cup \bigcup \{\mathcal{N}_p \cap X \mid X \in \mathcal{X}\} = \mathcal{N}_p \cap N = \mathcal{Y} \cup \bigcup \{\mathcal{N}_p \cap Y \mid Y \in \mathcal{Y}\}.$$

Let $X \in \mathcal{X}$. Suffice to get $X \in \mathcal{Y}$. Suppose to the contrary $X \notin \mathcal{Y}$. Then $X \in \mathcal{N}_p \cap Y$ for some $Y \in \mathcal{Y}$. Then $X <_{\omega_1} Y$ and so it is impossible to have $Y \in \mathcal{X} \cup \bigcup \{\mathcal{N}_p \cap X' \mid X' \in \mathcal{X}\}$. This is absurd.

Definition. For $p, q \in P$, let $q \leq p$ in P, if

- $\mathcal{N}_q \supseteq \mathcal{N}_p$.
- (freeze) For any $N \in \mathcal{N}_p$, there exists $Y \in (\mathcal{N}_q \cap N) \cup \{N\}$ s.t. $\mathcal{X}_p^N = \mathcal{X}_q^Y$.

Lemma. $(P, \leq)$ is a poset.

Proof. (ref) Let $p \in P$. Then $p \leq p$ in P.

(transitive) Let $r \leq q \leq p$ in P. Let $N \in \mathcal{N}_p$. Suffice to find $N_2 \in (\mathcal{N}_r \cap N) \cup \{N\}$ s.t. $\mathcal{X}_p^N = \mathcal{X}_r^{N_2}$. Since $q \leq p$ in P, there exists $N_1 \in (\mathcal{N}_q \cap N) \cup \{N\}$ s.t. $\mathcal{X}_p^N = \mathcal{X}_q^{N_1}$. Since $r \leq q$ in P, there exists $N_2 \in (\mathcal{N}_r \cap N_1) \cup \{N_1\}$ s.t. $\mathcal{X}_q^{N_1} = \mathcal{X}_r^{N_2}$. Hence, we have $\mathcal{X}_p^N = \mathcal{X}_r^{N_2}$ and $N_2 \in (\mathcal{N}_r \cap N_1) \cup \{N_1\} \subseteq (\mathcal{N}_r \cap N) \cup \{N\}$.

Lemma. (Dense) (1) For $p \in P$ and $x \in H_{\omega_2}$, there exists $q \in P$ s.t. $q \leq p$ in P and $x \in N_q$.

(2) Let (θ, N^*, P) be as usual. Let $p \in P \cap N^*$. Then there exists $q \in P$ s.t. $q \leq p$ in P and $N := H_{\omega_2} \cap N^* \in \mathcal{N}_q$.

Proof. (1) Let $W \in C_0$ with $\{p, x\} \in W$. Let $q = \mathcal{N}_p \cup \{W\}$. Then $q \in P$, $q \leq p$ in P, and $x \in N_q = W$.
(2) We have $N := H_{\omega_2} \cap N^* \in C_0$ and $p \in P \cap N$. Let $q = \mathcal{N}_p \cup \{N\}$. Then $q \in P$, $q \leq p$ in P and $N \in \mathcal{N}_q$.

Lemma. Let (θ, N^*, P) be as usual. Let $p \in P$ s.t. $N := H_{\omega_2} \cap N^* \in \mathcal{N}_p$. Then p is (P, N^*)-strongly-generic. By this we mean that for any $D \subseteq P \cap N^*$ that is predense in $P \cap N^*$, we have D is predense below p in P.

Proof. Let $D \subseteq P \cap N^* = P \cap N$ be predense in $P \cap N$. Want D is predense below p in P. To show this, let us take an arbitrary extension of p in P. Let us denote this extension by p again. Hence, $N \in \mathcal{N}_p$.
Since $N = \bigcup (C_0 \cap N)$, take $W \in C_0 \cap N$ s.t. $\mathcal{N}_p \cap N \in W$. Let $r = (\mathcal{N}_p \cap N) \cup \{W\}$. Then $r \in P \cap N$. Take $q_0 \in P \cap N$ and $d \in D$ s.t. $q_0 \leq r, d$ in $P \cap N$. Hence,

- For any $X \in \mathcal{N}_p \cap N$, there exists $Y_1 \in (\mathcal{N}_{q0} \cap X) \cup \{X\}$ s.t. $\mathcal{X}_p^X = \mathcal{X}_{q0}^{Y_1}$.

Let $q = \mathcal{N}_p \cup \{\phi_{NN'}(Z) \mid N' \in \mathcal{N}_p,\ N =_{\omega_1} N',\ Z \in \mathcal{N}_{q0}\}$. Then we know q satisfies (el with top), (iso), (up), (down), and (pre-partition). Hence, $q \in P$. Need to observe $q \leq p$, q_0 in P. We have

$\mathcal{N}_q \supseteq \mathcal{N}_p, \mathcal{N}_{q_0}$.

- If $X \in \mathcal{N}_p$ with $N <_{\omega_1} X$, then $\mathcal{X}_p^X = \mathcal{N}_q^X$.
- $\mathcal{X}_p^N = \mathcal{X}_r^W = \mathcal{X}_{q0}^{Y_1} = \mathcal{N}_q^{Y_1}$ for some $Y_1 \in (\mathcal{N}_{q_0} \cap W) \cup \{W\} \subseteq \mathcal{N}_q \cap N \subseteq (\mathcal{N}_q \cap N) \cup \{N\}$.
- (copying) If $N' \in \mathcal{N}_p$ s.t. $N' =_{\omega_1} N$, then $\mathcal{X}_p^{N'} = \phi_{NN'}[\mathcal{X}_p^N] = \phi_{NN'}[\mathcal{X}_q^{Y_1}] = \mathcal{X}_q^{\phi_{NN'}(Y_1)}$, $\phi_{NN'}(Y_1) \in (\mathcal{N}_q \cap N') \cup \{N'\}$.
- If $X \in \mathcal{N}_p \cap N$, then $\mathcal{X}_p^X = \mathcal{X}_r^X = \mathcal{X}_{q0}^{Y_1} = \mathcal{X}_q^{Y_1}$ for some $Y_1 \in (\mathcal{N}_{q_0} \cap X) \cup \{X\} \subseteq (\mathcal{N}_q \cap X) \cup \{X\}$.
- (copying) If $X' \in \mathcal{N}_p$, $X \in \mathcal{N}_p \cap N$, and $X' =_{\omega_1} X$, then $\mathcal{X}_p^{X'} = \phi_{XX'}[\mathcal{X}_p^X] = \phi_{XX'}[\mathcal{X}_q^{Y_1}] = \mathcal{X}_q^{\phi_{XX'}[Y_1]}$, and $\phi_{XX'}[Y_1] \in (\mathcal{N}_q \cap X') \cup \{X'\}$.

Hence, $q \leq p$ in P.

- For $X \in \mathcal{N}_{q_0}$, we have $\mathcal{N}_q \cap X = \mathcal{N}_{q_0} \cap X$ and so $\mathcal{X}_{q_0}^X = \mathcal{X}_q^X$.

Hence, $q \leq q_0$ in P.

Lemma. (CH) P has the ω_2-cc.

Proof. Let $\langle p_i \mid i < \omega_2 \rangle$ be indexed conditions of P. For each $i < \omega_2$, let $N_i \in C_0$ s.t. $\{i, p_i\} \subseteq N_i$. By CH, we may assume the N_i forms a delta system. For $i < j$, we may further assume that $\Delta := (\omega_2 \cap N_i) \cap (\omega_2 \cap N_j) < (\omega_2 \cap N_i) \setminus \Delta < (\omega_2 \cap N_j) \setminus \Delta \neq \emptyset$ and there exists the isomorphism $\phi_{N_iN_j}: (N_i, \in, p_i) \longrightarrow (N_j, \in, p_j)$ s.t. $\phi_{N_iN_j}(x) = x$ for any $x \in N_i \cap N_j$. Fix two $i < j$. Let $N \in C_0$ s.t. $\{N_i, N_j\} \in N$. Let $\mathcal{N}_q = \mathcal{N}_{p_i} \cup \mathcal{N}_{p_j} \cup \{N_i, N_j, N\}$. Then $\mathcal{N}_q \cap N_i = \mathcal{N}_{p_i}$ and $\mathcal{N}_q \cap N_j = \mathcal{N}_{p_j}$. Hence, $q \in P$ and $q \leq p_i, p_j$ in P.

Lemma. Let G be P-generic over V. In $V[G]$, form

$$\mathcal{N} = \bigcup G.$$

Then $\mathcal{N}$ satisfies (el), (iso), (up), (down), (2-partition), and (cof).

Proof. (2-partition) Let $N \in \mathcal{N}$.

Case. $\{\alpha_X \mid X \in \mathcal{N} \cap N\}$ had no last element: We show $N \in \mathrm{dir}(\mathcal{N})$. To this end, let $\{X_1, X_2\} \subseteq \mathcal{N} \cap N$. Want to find $X_0 \in \mathcal{N} \cap N$ s.t. $\{X_1, X_2\} \subseteq X_0$. Let $\xi = \sup\{\alpha_X \mid X \in \mathcal{N} \cap N\}$. Take $p \in G$ s.t. $\{X_1, X_2, N\} \subseteq \mathcal{N}_p$. Then $\{X_1, X_2\} \subseteq \mathcal{X}_p^N \cup \bigcup\{\mathcal{N}_p \cap X \mid X \in \mathcal{X}_p^N\}$, $\mathcal{X}_p^N <_{\omega_1} \xi$ and so there exists $N_2 \in \mathcal{N} \cap N$ s.t. $\mathcal{X}_p^N <_{\omega_1} N_2$.
Take $q \in G$ s.t. $q \leq p$ and $N_2 \in \mathcal{N}_q$. Since $q \leq p$ and $N \in \mathcal{N}_p$, there exists $X_0 \in (\mathcal{N}_q \cap N) \cup \{N\}$ s.t. $\mathcal{X}_p^N = \mathcal{X}_q^{X_0}$. Then $X_0 \leq_{\omega_1} N_2$. Hence $\{X_1, X_2\} \subseteq \mathcal{X}_q^{X_0} \cup \bigcup\{\mathcal{N}_p \cap X \mid X \in \mathcal{X}_q^{X_0}\} \subseteq X_0 \in \mathcal{N} \cap N$.

Case. $\{\alpha_X \mid X \in \mathcal{N} \cap N\}$ had the last element α^*: We show $N \in \mathrm{suc}(\mathcal{N})$. Let

$$\mathcal{X} = \{X \in \mathcal{N} \cap N \mid \alpha_X = \alpha^*\}.$$

Let us fix any $X_0 \in \mathcal{X}$. Let $p \in G$ s.t. $\{N, X_0\} \subseteq \mathcal{N}_p$.

Claim. $\mathcal{X} = \mathcal{X}_p^N$.

Proof. Since $X_0 \in \mathcal{X}_p^N$, we have $\mathcal{X} \supseteq \mathcal{X}_p^N$. Conversely, we show $\mathcal{X} \subseteq \mathcal{X}_p^N$. Let $Y \in \mathcal{X}$. Take $q \in G$ s.t.

$q \le p$ and $Y \in \mathcal{N}_q$. Since $\{N, Y\} \subseteq \mathcal{N}_q$, we must have $Y \in \mathcal{X}^N_q$. By (freeze), we have $Z \in (\mathcal{N}_q \cap N) \cup \{N\}$ s.t. $\mathcal{X}^N_p = \mathcal{X}^Z_q$ However, we must have $Z = N$. Hence, $Y \in \mathcal{X}^N_p$.

Claim. $\mathcal{N} \cap N = \mathcal{X}^N_p \cup \bigcup\{\mathcal{N} \cap X \mid X \in \mathcal{X}^N_p\}$ and so $N \in \mathrm{suc}(\mathcal{N})$.

Proof. It suffices to show $\subseteq$. Let $A \in \mathcal{N} \cap N$. Take $q \in G$ s.t. $q \le p$ in P and $A \in \mathcal{N}_q$. We first observe $\mathcal{X}^N_p = \mathcal{X}^N_q$. Since $q \le p$ in P, we have $\mathcal{X}^N_p = \mathcal{X}^Y_q$ for some $Y \in (\mathcal{N}_q \cap N) \cup \{N\}$. Since $\alpha^* < \alpha_Y \le \alpha_N$, we must have $\alpha_Y = \alpha_N$ and so $Y = N$. Hence $\mathcal{X}^N_p = \mathcal{X}^N_q$. Then

$$A \in \mathcal{N}_q \cap N = \mathcal{X}^N_q \cup \bigcup\{\mathcal{N}_q \cap X \mid X \in \mathcal{X}^N_q\} \subseteq \mathcal{X}^N_p \cup \bigcup\{\mathcal{N} \cap X \mid X \in \mathcal{X}^N_p\}.$$

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