Generation of a Sparse Control Input Optimal in the Infinite Horizon*

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Sparse optimal control is considered in the infinite horizon. In the literature, sparse control has been considered mostly in a finite horizon since the associated optimization problem is difficult to solve directly in the infinite horizon. It is shown in this report that an optimal solution of the infinite-horizon sparse control problem can be obtained through a solution of a finite-horizon problem. This is due to sparsity of the optimal solution in the sense that the optimal control input is constantly equal to zero at its tail. This result is not only useful in practice but also notable in theory because a similar phenomenon does not occur in the traditional optimal control with the 2-norm.

Keywords: sparse control, optimal control, infinite-horizon control problem, optimality condition, anti-stable part.

1. Introduction

Sparse control is a control method for saving energy and achieves a control objective with an input equal to zero for a long time duration. Such a control input can be obtained by minimizing the sum of the 1-norm of the input in a finite horizon when the discrete-time framework is used. In many control applications, however, it is difficult to fix beforehand a finite horizon where a control objective is achieved and thus the infinite horizon is preferable. A problem here is that it is not clear how the sum of the 1-norm can be minimized in the infinite horizon. Although the technique of the self-triggered control or the model-predictive control is used in the literature [1, 2], optimality in the infinite horizon is not guaranteed in this case.

In this report, it is shown that a control input optimal in some finite-horizon problem is actually optimal in the infinite horizon as well if it is extended by zero input at its tail. This result is practically important because it enables us to generate an optimal input in the infinite horizon by solving a finite-horizon problem. This is due to a sparse property of the optimal input caused by the use of the 1-norm. Since a similar phenomenon does not occur in the traditional optimal control with the 2-norm, the result is notable also from a theoretical point of view.

In [3], a group of the present author considered generation of a sparse optimal input with model-predictive control. The result of this report gives a theoretical foundation to the result there.

The following notation is used. For a vector x, the symbol $\|x\|_1$ stands for its 1-norm, *i.e.*, the sum of the absolute values of the components of the vector. The symbol $\|x\|_\infty$ indicates the ∞-norm of the vector x, *i.e.*, the maximum of the absolute values of its components. For a matrix A, the symbol $\|A\|_1$ denotes its norm induced by the 1-norm of a vector. The symbols I and O stand for the identity matrix and the zero matrix of appropriate size, respectively. For a real number V, the smallest integer larger than or equal to V is expressed by $\lceil V \rceil$.

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2. Result

A plant to be controlled is a discrete-time system:

$$x(k+1) = Ax(k) + Bu(k) \ \ (k = 0,\ 1,\ \ldots), \quad x(0) = \xi \tag{1}$$

with a state $x(k) \in \mathbb{R}^n$ and an input $u(k) = (u_1(k)\ \ u_2(k)\ \ \cdots\ \ u_m(k))^{\mathrm{T}} \in \mathbb{R}^m$. It is assumed that the matrix A does not have an eigenvalue on the unit circle in the complex plane and that (A, B) is controllable.

The infinite-horizon control problem considered in this report is the following:

$$\begin{aligned} P:\ \text{minimize}\ & \sum_{k=0}^{\infty} \|u(k)\|_1 \\ \text{subject to}\ & |u_i(k)| \le 1 \ \ (i = 1,\ 2,\ \ldots,\ m;\ \ k = 0,\ 1,\ \ldots), \\ & x(k+1) = Ax(k) + Bu(k) \ \ (k = 0,\ 1,\ \ldots), \\ & x(0) = \xi, \quad x(k) \to 0 \ \ (k \to \infty). \end{aligned}$$

The formulation of the problem is basically the same as a standard formulation of the sparse control [1, 2]. Indeed, the objective function is the sum of the 1-norm of the input in order to induce a sparse property of the input, *i.e.*, $u(k) = 0$ for many k's. The difference is that, while the problem has been considered in a finite horizon in the literature, the problem P is in the infinite horizon. When the time duration for control is difficult to fix beforehand, the infinite-horizon problem P appears more acceptable. A problem here is that it is not clear how to solve the infinite-horizon problem unlike the finite-horizon counterpart. In the following, we focus on the input $u(k)$ as a variable of the problem P. This is because the state $x(k)$ is uniquely determined once the input $u(k)$ is provided.

The claim of this report is that an optimal solution of the problem P can be obtained by solving some finite-horizon problem. The considered finite-horizon problem is introduced next.

With an invertible matrix T, we decompose the dynamics of the plant into its stable part and anti-stable part. That is, we modify the representation (1) into the form

$$\begin{pmatrix} x_{\mathrm{s}}(k+1) \\ x_{\mathrm{a}}(k+1) \end{pmatrix} = \begin{pmatrix} A_{\mathrm{s}} & O \\ O & A_{\mathrm{a}} \end{pmatrix} \begin{pmatrix} x_{\mathrm{s}}(k) \\ x_{\mathrm{a}}(k) \end{pmatrix} + \begin{pmatrix} B_{\mathrm{s}} \\ B_{\mathrm{a}} \end{pmatrix} u(k) \ \ (k = 0,\ 1, \ldots), \quad \begin{pmatrix} x_{\mathrm{s}}(0) \\ x_{\mathrm{a}}(0) \end{pmatrix} = T^{-1}\xi$$

by the state transform $x(k) = T(x_{\mathrm{s}}(k)^{\mathrm{T}}\ \ x_{\mathrm{a}}(k)^{\mathrm{T}})^{\mathrm{T}}$, where A_{s} has all of its eigenvalues inside the unit circle and A_{a} outside the unit circle. The subsystem $(A_{\mathrm{s}}, B_{\mathrm{s}})$ is called the stable part of the plant and $(A_{\mathrm{a}}, B_{\mathrm{a}})$ the anti-stable part. It is possible that either the stable part or the anti-stable part does not exist. When the anti-stable part does not exist, the optimal solution of the problem P is obviously the zero input $u(k) = 0$ $(k = 0,\ 1,\ \ldots)$. The existence of the anti-stable part will be assumed henceforth. It is possible to assume $\|A_{\mathrm{a}}^{-1}\|_1 < 1$ by an appropriate choice of T (See Lemma 5.6.10 of [4]). The dimension of $x_{\mathrm{a}}(k)$, the anti-stable part of the state, is denoted by n_{a}.

We consider the following finite-horizon problem for a positive integer N:

$$F:\ \text{minimize}\ \sum_{k=0}^{N-1} \|u(k)\|_1$$

$$\text{subject to } |u_i(k)| \le 1 \ \ (i = 1, 2, \dots, m;\ \ k = 0, 1, \dots, N-1),$$
$$x_a(k+1) = A_a x_a(k) + B_a u(k) \ \ (k = 0, 1, \dots, N-1),$$
$$x_a(0) = (O \ \ I)T^{-1}\xi, \quad x_a(N) = 0.$$

Note in the problem F only the anti-stable part of the plant is considered. The positive integer N is referred to as the horizon length. A solution of the problem F is an input $u(k)$ of length N defined for $k = 0, 1, \dots, N-1$. This input can be extended by adding the zero input $u(k) = 0$ for $k = N, N+1, \dots$. The result of this report is that an optimal solution of the infinite-horizon problem P can be obtained by such zero extension.

Theorem. *For a large enough horizon length N, the zero extension of an optimal solution of the problem F is optimal in the problem P.*

Based on this theorem, we can actually produce an optimal solution of the infinite-horizon problem P by solving the finite-horizon problem F. As will be seen in the following, sparsity of the optimal solution plays an important role for its proof.

3. Proof

On the finite-horizon problem F, the following optimality condition is available. Here we write the ith column of B_a, the anti-stable part of the B matrix, as b_i for $i = 1, 2, \dots, m$.

Lemma 1. *The input $u^*(k)$ $(k = 0, 1, \dots, N-1)$ is optimal in the problem F if and only if there exist a state $x_a^*(k)$ $(k = 0, 1, \dots, N)$ and a costate $p_a^*(k)$ $(k = 1, \dots, N)$ satisfying*

$$x_a^*(k+1) = A_a x_a^*(k) + B_a u^*(k) \ \ (k = 0, 1, \dots, N-1), \tag{2}$$
$$x_a^*(0) = (O \ \ I)T^{-1}\xi, \quad x_a^*(N) = 0, \tag{3}$$
$$p_a^*(k)^{\mathrm{T}} = p_a^*(k+1)^{\mathrm{T}} A_a \ \ (k = 1, 2, \dots, N-1), \tag{4}$$
$$u_i^*(k) = \mathop{\arg\min}_{|u_i(k)|\le 1}\Big[|u_i(k)| + p_a^*(k+1)^{\mathrm{T}} b_i u_i(k)\Big] \ \ (i = 1, 2, \dots, m;\ k = 0, 1, \dots, N-1). \tag{5}$$

The input $u_i^*(k)$ satisfying the last equation has the following properties:

$$p_a^*(k+1)^{\mathrm{T}} b_i < -1 \quad \textit{implies} \quad u_i^*(k) = 1;$$
$$p_a^*(k+1)^{\mathrm{T}} b_i = -1 \quad \textit{implies} \quad 0 \le u_i^*(k) \le 1;$$
$$-1 < p_a^*(k+1)^{\mathrm{T}} b_i < 1 \quad \textit{implies} \quad u_i^*(k) = 0;$$
$$p_a^*(k+1)^{\mathrm{T}} b_i = 1 \quad \textit{implies} \quad -1 \le u_i^*(k) \le 0;$$
$$p_a^*(k+1)^{\mathrm{T}} b_i > 1 \quad \textit{implies} \quad u_i^*(k) = -1.$$

Proof. Necessity follows from Section IV-D of [1] or Proposition 3.3.2 of [5].

Sufficiency follows from Proposition 3.3.4 of [6]. For completeness, the proof is presented in a form adapted to the present context. Let $u^*(k)$, $x_a^*(k)$, and $p_a^*(k)$ satisfy the conditions (2)–(5). For any $u(k)$ and $x_a(k)$ that satisfy

$$|u_i(k)| \le 1, \ \ x_a(k+1) = A_a x_a(k) + B_a u(k), \ \ x_a(0) = (O \ \ I)T^{-1}\xi, \ \ x_a(N) = 0,$$

we have

$$\sum_{k=0}^{N-1}\|u(k)\|_1 = \sum_{k=0}^{N-1}\|u(k)\|_1 + \sum_{k=0}^{N-1} p_{\mathrm{a}}^*(k+1)^{\mathrm{T}}\Big[A_{\mathrm{a}}x_{\mathrm{a}}(k) + B_{\mathrm{a}}u(k) - x_{\mathrm{a}}(k+1)\Big]$$
$$= \sum_{k=0}^{N-1}\sum_{i=1}^{m}\Big[|u_i(k)| + p_{\mathrm{a}}^*(k+1)^{\mathrm{T}}b_iu_i(k)\Big] + \sum_{k=1}^{N-1}\Big[p_{\mathrm{a}}^*(k+1)^{\mathrm{T}}A_{\mathrm{a}} - p_{\mathrm{a}}^*(k)^{\mathrm{T}}\Big]x_{\mathrm{a}}(k)$$
$$+ p_{\mathrm{a}}^*(1)^{\mathrm{T}}A_{\mathrm{a}}x_{\mathrm{a}}(0) - p_{\mathrm{a}}^*(N)^{\mathrm{T}}x_{\mathrm{a}}(N).$$

The first equality follows from $x_{\mathrm{a}}(k+1) = A_{\mathrm{a}}x_{\mathrm{a}}(k) + B_{\mathrm{a}}u(k)$ and the second from the change of the summation order. In the last expression, the first term is minimized at $u^*(k)$ due to (5) and the remaining terms are constant irrespective of $x_{\mathrm{a}}(k)$ due to (4). Therefore, it is minimized at $u^*(k)$.

The properties of $u_i^*(k)$ in the second statement easily follow from the condition (5). □

An optimal solution of the finite-horizon problem F is sparse as is shown next.

Lemma 2. *There exists some positive integer K depending on the anti-stable part of the plant $(A_{\mathrm{a}}, B_{\mathrm{a}})$ and its initial state $(O\ \ I)T^{-1}\xi$ such that any optimal solution of the finite-horizon problem F satisfies $u^*(k) = 0$ for $k = K,\ K+1,\ \ldots,\ N-1$ if the horizon length N is larger than K. Moreover, if we extend this optimal input to the length $N' > N$ by setting $u^*(k) = 0$ for $k = N,\ N+1,\ \ldots,\ N'-1$, this extended input is optimal in the problem F with the horizon length replaced by N'.*

Proof. Choose the horizon length N so that the problem F is feasible and let V be the objective function value of some feasible solution. Let k_0 be a positive integer larger than or equal to this N and satisfying $k_0 \geq n_{\mathrm{a}}(\lceil V\rceil + 1)$. On the other hand, note controllability of (A, B) implies controllability of $(A_{\mathrm{a}}, B_{\mathrm{a}})$. Since the controllability matrix $(B_{\mathrm{a}}\ \ A_{\mathrm{a}}B_{\mathrm{a}}\ \ \cdots\ \ A_{\mathrm{a}}^{n_{\mathrm{a}}-1}B_{\mathrm{a}})$ has a full row rank, there exists $g > 0$ such that $\|p^{\mathrm{T}}(B_{\mathrm{a}}\ \ A_{\mathrm{a}}B_{\mathrm{a}}\ \ \cdots\ \ A_{\mathrm{a}}^{n_{\mathrm{a}}-1}B_{\mathrm{a}})\|_\infty \geq \|p\|_\infty g$ for any vector p. Indeed, the corresponding property is well-known for the 2-norm and the 2-norm and the ∞-norm are equivalent in a finite-dimensional vector space. With this g, let k_1 be a nonnegative integer such that $g > \|A_{\mathrm{a}}^{-1}\|_1^{k_1+1}\|b_i\|_1$ for any $i = 1,\ 2,\ \ldots,\ m$. Such a k_1 exists due to $\|A_{\mathrm{a}}^{-1}\|_1 < 1$. Define K by $k_0 + k_1$.

For any $N > K$, we consider the problem F and its optimal solution $u^*(k)$ $(k = 0,\ 1,\ \ldots,\ N-1)$. This N is larger than the N considered above and the feasible solution considered there gives a feasible solution of the present problem F by zero extension. Hence, the present optimal solution $u^*(k)$ has the objective function value smaller than or equal to V.

We first show the existence of an integer $k_2 \leq k_0$ such that $|u_i^*(k)| < 1$ for any $i = 1,\ 2,\ \ldots, m$ and any $k = k_2-1,\ k_2-2,\ \ldots,\ k_2-n_{\mathrm{a}}$. Indeed, among $\lceil V\rceil + 1$ intervals $k_0 - 1 \geq k \geq k_0 - n_{\mathrm{a}}$, $k_0 - n_{\mathrm{a}} - 1 \geq k \geq k_0 - 2n_{\mathrm{a}}$, $\ldots$, $k_0 - \lceil V\rceil n_{\mathrm{a}} - 1 \geq k \geq k_0 - (\lceil V\rceil + 1)n_{\mathrm{a}}$, at least one interval should have the property above. If this is not the case, each interval has k such that $|u_i^*(k)| \geq 1$ for some i, which means $\|u^*(k)\|_1 \geq 1$. Then, the value of the objective function satisfies $\sum_{k=0}^{N-1}\|u^*(k)\|_1 \geq \lceil V\rceil + 1 > V$, which is a contradiction.

For the k_2 above, Lemma 1 implies

$$|p_{\mathrm{a}}^*(k+1)^{\mathrm{T}}b_i| \leq 1 \quad (i = 1,\ 2,\ \ldots, m;\ \ k = k_2-1,\ k_2-2,\ \ldots,\ k_2-n_{\mathrm{a}}).$$

Noting $p_{\mathrm{a}}^*(k_2)^{\mathrm{T}}A_{\mathrm{a}} = p_{\mathrm{a}}^*(k_2-1)^{\mathrm{T}}$, $p_{\mathrm{a}}^*(k_2-1)^{\mathrm{T}}A_{\mathrm{a}} = p_{\mathrm{a}}^*(k_2-2)^{\mathrm{T}}$, $\ldots$, we have

$$|p_{\mathrm{a}}^*(k_2)^{\mathrm{T}}b_i| \leq 1,\ \ |p_{\mathrm{a}}^*(k_2)^{\mathrm{T}}A_{\mathrm{a}}b_i| \leq 1,\ \ \ldots,\ \ |p_{\mathrm{a}}^*(k_2)^{\mathrm{T}}A_{\mathrm{a}}^{n_{\mathrm{a}}-1}b_i| \leq 1$$

for any $i = 1, 2, \ldots, m$. This leads to

$$1 \geq \left\| p_{\mathrm{a}}^*(k_2)^{\mathrm{T}} (B_{\mathrm{a}} \;\; A_{\mathrm{a}} B_{\mathrm{a}} \;\; \cdots \;\; A_{\mathrm{a}}^{n_{\mathrm{a}}-1} B_{\mathrm{a}}) \right\|_{\infty}.$$

By the definition of the positive number g, we have $1 \geq \|p_{\mathrm{a}}^*(k_2)\|_{\infty} g$.

Now for any integer k such that $K = k_0 + k_1 \leq k \leq N - 1$ and any $i = 1, 2, \ldots, m$,

$$|p_{\mathrm{a}}^*(k+1)^{\mathrm{T}} b_i| = |p_{\mathrm{a}}^*(k_2)^{\mathrm{T}} A_{\mathrm{a}}^{-(k-k_2+1)} b_i| \leq \|p_{\mathrm{a}}^*(k_2)\|_{\infty} \|A_{\mathrm{a}}^{-1}\|_1^{k-k_2+1} \|b_i\|_1 \leq \frac{1}{g} \|A_{\mathrm{a}}^{-1}\|_1^{k-k_2+1} \|b_i\|_1.$$

Since $k - k_2 + 1 \geq k - k_0 + 1 \geq k_1 + 1$, the above value is less than 1 and thus $u_i^*(k) = 0$ for any $i = 1, 2, \ldots, m$.

In order to show the second statement, extend the state and the costate to the length N' so that

$$x_{\mathrm{a}}^*(k+1) = A_{\mathrm{a}} x_{\mathrm{a}}^*(k) + B_{\mathrm{a}} u^*(k) \quad (k = N, N+1, \ldots, N'-1),$$
$$p_{\mathrm{a}}^*(k)^{\mathrm{T}} = p_{\mathrm{a}}^*(k+1)^{\mathrm{T}} A_{\mathrm{a}} \quad (k = N, N+1, \ldots, N'-1).$$

Since these extended input, state, and costate satisfy the optimality condition in Lemma 1, the extended input $u^*(k)$ is optimal in the problem F with the horizon length N'. □

Having the optimal solution of the finite-horizon problem F for some $N > K$ and extending it by the zero input, we can obtain a feasible solution of the infinite-horizon problem P. To see this, note that $x_{\mathrm{a}}^*(N) = 0$ and $u^*(k) = 0$ for $k = N, N+1, \ldots$, which implies the state $x_{\mathrm{a}}^*(k)$ remains zero after $k = N$. On the other hand, the state of the stable part, $x_{\mathrm{s}}^*(k)$, approaches zero after $k = N$ due to the zero input. For the moment, it is not clear whether the input obtained like this is optimal or not in the infinite-horizon problem P. We will consider this optimality from now on.

Among feasible inputs for the problem P, we need to focus only on inputs of finite support.

Lemma 3. *Let $u(k)$ $(k = 0, 1, \ldots)$ be any feasible solution of the infinite-horizon problem P having a finite objective function value. Then, there exists another feasible solution $\overline{u}(k)$ $(k = 0, 1, \ldots)$ that satisfies $\overline{u}(k) = 0$ $(k = K_u, K_u + 1, \ldots)$ for some nonnegative integer K_u and makes the objective function value smaller than or equal to that of $u(k)$.*

Proof. The assumed controllability of (A, B) implies the controllability of the anti-stable part $(A_{\mathrm{a}}, B_{\mathrm{a}})$. The corresponding controllability matrix $W_{\mathrm{a}} = (B_{\mathrm{a}} \;\; A_{\mathrm{a}} B_{\mathrm{a}} \;\; \cdots \;\; A_{\mathrm{a}}^{n_{\mathrm{a}}-1} B_{\mathrm{a}})$ hence has a full row rank. Since the input $u(k)$ has a finite objective function value, $u(k) \to 0$ as $k \to \infty$. This implies the existence of $\overline{k}$ such that $|u_i(k)| \leq 1/2$ for any $i = 1, 2, \ldots, m$ and any $k = \overline{k}, \overline{k}+1, \ldots, \overline{k}+n_{\mathrm{a}}-1$. Now choose a positive integer $K_u > \overline{k} + n_{\mathrm{a}} - 1$ so that each element of the vector defined by

$$(\delta_{n_{\mathrm{a}}-1}^{\mathrm{T}} \;\; \delta_{n_{\mathrm{a}}-2}^{\mathrm{T}} \;\; \cdots \;\; \delta_0^{\mathrm{T}})^{\mathrm{T}} = -W_{\mathrm{a}}^{\mathrm{T}} (W_{\mathrm{a}} W_{\mathrm{a}}^{\mathrm{T}})^{-1} A_{\mathrm{a}}^{-(K_u - \overline{k} - n_{\mathrm{a}})} x_{\mathrm{a}}(K_u)$$

has a magnitude less than or equal to $1/2$ and

$$\|W_{\mathrm{a}}^{\mathrm{T}} (W_{\mathrm{a}} W_{\mathrm{a}}^{\mathrm{T}})^{-1}\|_1 \|A_{\mathrm{a}}^{-1}\|_1^{K_u - \overline{k} - n_{\mathrm{a}}} \leq \frac{1}{\|B_{\mathrm{a}}\|_1}.$$

Such a K_u exists because the matrix A_{a}^{-1} has its eigenvalues inside the unit circle and satisfies $\|A_{\mathrm{a}}^{-1}\|_1 < 1$.

Define a new input $\overline{u}(k)$ as follows:

$$\overline{u}(k) = \begin{cases} u(k) + \delta_j & \text{for } k = \overline{k} + j \ \ (j = 0,\ 1,\ \ldots,\ n_{\mathrm{a}} - 1), \\ 0 & \text{for } k = K_u,\ K_u + 1,\ \ldots, \\ u(k) & \text{otherwise.} \end{cases}$$

Recall that each element of δ_j has the magnitude less than or equal to $1/2$, which means $|\overline{u}_i(k)| \leq 1$ for any i and any k. When the input $\overline{u}(k)$ is applied to the plant, the corresponding state is written as $\overline{x}(k)$ and its stable part and anti-stable part are as $\overline{x}_{\mathrm{s}}(k)$ and $\overline{x}_{\mathrm{a}}(k)$, respectively. Here we have

$$\begin{aligned} \overline{x}_{\mathrm{a}}(K_u) &= A_{\mathrm{a}}^{K_u} x_{\mathrm{a}}(0) + \sum_{k=0}^{K_u-1} A_{\mathrm{a}}^{K_u-k-1} B_{\mathrm{a}} \overline{u}(k) \\ &= A_{\mathrm{a}}^{K_u} x_{\mathrm{a}}(0) + \sum_{k=0}^{K_u-1} A_{\mathrm{a}}^{K_u-k-1} B_{\mathrm{a}} u(k) + \sum_{j=0}^{n_{\mathrm{a}}-1} A_{\mathrm{a}}^{K_u-\overline{k}-j-1} B_{\mathrm{a}} \delta_j \\ &= x_{\mathrm{a}}(K_u) + A_{\mathrm{a}}^{K_u-\overline{k}-n_{\mathrm{a}}} (B_{\mathrm{a}} \ \ A_{\mathrm{a}} B_{\mathrm{a}} \ \ \cdots \ \ A_{\mathrm{a}}^{n_{\mathrm{a}}-1} B_{\mathrm{a}}) (\delta_{n_{\mathrm{a}}-1}^{\mathrm{T}} \ \ \delta_{n_{\mathrm{a}}-2}^{\mathrm{T}} \ \ \cdots \ \ \delta_0^{\mathrm{T}})^{\mathrm{T}} \\ &= 0. \end{aligned}$$

After $k = K_u$ the anti-stable part of the state, $\overline{x}_{\mathrm{a}}(k)$, constantly equals to zero because so does the input $\overline{u}(k)$. The stable part of the state, $\overline{x}_{\mathrm{s}}(k)$, converges to zero after $k = K_u$ again by the zero input. Hence the whole state $\overline{x}(k)$ converges to zero, which means that the input $\overline{u}(k)$ is feasible in the problem P.

It remains to show that $\overline{u}(k)$ does not make the objective function value larger than $u(k)$. For that, we need to show

$$\|(\delta_{n_{\mathrm{a}}-1}^{\mathrm{T}} \ \ \delta_{n_{\mathrm{a}}-2}^{\mathrm{T}} \ \ \cdots \ \ \delta_0^{\mathrm{T}})^{\mathrm{T}}\|_1 \leq \sum_{k=K_u}^{\infty} \|u(k)\|_1.$$

Note first that

$$\begin{aligned} \|(\delta_{n_{\mathrm{a}}-1}^{\mathrm{T}} \ \ \delta_{n_{\mathrm{a}}-2}^{\mathrm{T}} \ \ \cdots \ \ \delta_0^{\mathrm{T}})^{\mathrm{T}}\|_1 &\leq \|W_{\mathrm{a}}^{\mathrm{T}} (W_{\mathrm{a}} W_{\mathrm{a}}^{\mathrm{T}})^{-1}\|_1 \|A_{\mathrm{a}}^{-1}\|_1^{K_u-\overline{k}-n_{\mathrm{a}}} \|x_{\mathrm{a}}(K_u)\|_1 \\ &\leq \frac{1}{\|B_{\mathrm{a}}\|_1} \|x_{\mathrm{a}}(K_u)\|_1. \end{aligned} \tag{6}$$

On the other hand, for any $\ell > K_u$,

$$x_{\mathrm{a}}(\ell) = A_{\mathrm{a}}^{\ell-K_u} x_{\mathrm{a}}(K_u) + \sum_{k=K_u}^{\ell-1} A_{\mathrm{a}}^{\ell-k-1} B_{\mathrm{a}} u(k),$$

which implies

$$A_{\mathrm{a}}^{K_u-\ell} x_{\mathrm{a}}(\ell) = x_{\mathrm{a}}(K_u) + \sum_{k=K_u}^{\ell-1} A_{\mathrm{a}}^{K_u-k-1} B_{\mathrm{a}} u(k).$$

In the limit of $\ell \to \infty$, $A_{\mathrm{a}}^{K_u-\ell}$ converges to zero and so does $x_{\mathrm{a}}(\ell)$, which gives

$$0 = x_{\mathrm{a}}(K_u) + \sum_{k=K_u}^{\infty} A_{\mathrm{a}}^{K_u-k-1} B_{\mathrm{a}} u(k).$$

Thus we have

$$\|x_{\mathrm{a}}(K_u)\|_1 \le \sum_{k=K_u}^{\infty} \|A_{\mathrm{a}}^{-1}\|_1^{k+1-K_u} \|B_{\mathrm{a}}\|_1 \|u(k)\|_1 \le \|B_{\mathrm{a}}\|_1 \sum_{k=K_u}^{\infty} \|u(k)\|_1. \tag{7}$$

The inequalities (6) and (7) give the desired inequality. □

Now we are ready to prove the theorem.

Proof of Theorem. Let $u(k)$ $(k = 0, 1, \ldots)$ be any feasible solution of the problem P having a finite objective function value. Lemma 3 implies the existence of another feasible solution $\bar{u}(k)$ $(k = 0, 1, \ldots)$ such that $\bar{u}(k) = 0$ for $k = K_u, K_u + 1, \ldots$ and the objective function value of $\bar{u}(k)$ is smaller than or equal to that of $u(k)$. Now we invoke Lemma 2 and consider an optimal solution of F for a large enough N. Then this optimal solution $u^*(k)$ $(k = 0, 1, \ldots, N-1)$ is sparse in the sense that $u(k) = 0$ for $k = K, K+1, \ldots, N-1$. Let N' be the maximum of N and K_u and extend $u^*(k)$ to the length N' with the zero input. If we compare $u^*(k)$ and $\bar{u}(k)$ in the interval $k = 0, 1, \ldots, N'-1$, the input $u^*(k)$ makes the objective function value smaller because it is optimal in the problem F with the horizon length N' due to Lemma 2. If we extend $u^*(k)$ to the infinite length, it is feasible in the problem P and makes the objective function value smaller than or equal to $\bar{u}(k)$ and thus $u(k)$. Since $u(k)$ is arbitrary, $u^*(k)$ is optimal in P. □

4. Conclusion

We consider in this report generation of a sparse control input optimal in the infinite horizon. The objective function here is the sum of the 1-norm of the input, which makes the optimal input sparse. Thanks to this property, the optimal input in some finite-horizon problem turns out to be optimal also in the infinite-horizon problem. This is considered not only useful in practice but also notable in theory because a similar phenomenon does not occur in the traditional optimal control with the 2-norm. As considered in [3] by a group of the present author, the property above can be used further for generation of a sparse optimal input with model-predictive control.

The proof of the result depends on special properties of the sparse control problem considered in [1, 2]. It is an interesting question whether the present result can be extended to a more general optimal control problem using the 1-norm. The research is proceeding in this direction.

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