Ikuta's Integrated Theory on Optimal Stopping Problems for Asset Selling and Buying: A Brief Review and Its Potential Application in Option Trading

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Abstract

This paper provides a brief review of Ikuta's integrated theory, delving into optimal stopping problems associated with asset selling and buying. His focus is on four types of decision-making problems that stem from determining whether the buyer or the seller proposes the transaction price in both asset selling and buying problems. Ikuta's exploration originates from his inquiries into the seemingly intuitive symmetrical and analogous relationships within the four types selling and buying problems. Notably, Ikuta uncovers that these relationships are not always straightforward. Furthermore, Ikuta introduces novel elements, including Quitting Penalty and Search-skipping Possibility. This results in the discovery of unique optimal decision rules customized for various asset trading models, successfully addressing aspects that are concealed from conventional analytic frameworks in optimal stopping problems. Specifically, these rules identify the Optimal Initiation Time for conducting search activities. Recognizing its potential impact on existing research, this paper aims to outline the key aspects of Ikuta's integrated theory and share insights derived from his study. In concluding remarks, the review suggests a prospective application of Ikuta's research to American options.

Keywords : Optimal stopping problem, Optimal decision rule, Optimal initiating time, Deadline falling

1. Introduction

Within the research field of the optimal stopping problem, the Asset Trading Problem focuses on deriving optimal decision rules that maximize profits for either the buyer or the seller during the process of buying and selling a specific asset, such as a car, house, stock, and so on. When examined from the standpoint of decision-making for the asset seller, it is referred to as the 'Asset Selling Problem.' Conversely, from the viewpoint of the asset buyer, it is termed the 'Asset Buying Problem.' The following two examples illustrate the nature of asset selling and buying problems.

- Asset Selling Problem

Suppose you are assigned to work overseas and need to sell your car by a deadline before departing from your home country. As the primary entity in this transaction, you must establish the selling price for potential buyers, who act as the counterparts in this transaction. Setting your price too low, e.g., below market value, would result in a swift sale, while setting it too high might discourage buyers, potentially leading to a last-minute sale at a reduced price as the deadline approaches. In these scenarios, determining the optimal selling price becomes a crucial task, involving an assessment of whether the price will attract a purchase or possibly lead to rejection.

On the other hand, another scenario exists. If you refrain from setting a price and allow potential buyers to propose their own prices in order to seek a buyer offering a favorable price for you, your decision-making changes. You face a distinct challenge: deciding whether to accept or reject a buyer's price based on your *minimum* permissible selling price, referred to as the 'reservation price.' This distinction illustrates how the seller's decision-making, as the primary entity, varies based on whether you set the selling price or allow buyers to propose their own.

- Asset Buying Problem

Similar dynamics apply in the asset buying problem. Suppose you need to buy a car for commuting at your overseas assignment office by a specific deadline after arriving in the country. When you visit a seller, if the price for the car you want to buy is low, you would likely proceed with the purchase. However, if the price is high, you might seek other sellers. Yet rushing to buy an expensive option too soon might lead to regret upon discovering better deals later. Conversely, tirelessly searching for a cheaper option could result in being forced to make a pricey purchase close to the deadline, especially when the previously found reasonably priced car has already been sold. In these scenarios, for every seller's price, you have to decide whether to accept or reject it based on your *maximum* permissible buying price, i.e., your 'reservation price.'

On the flip side, in a situation where you present your own buying price to a seller to find a potential seller offering a price as reasonable as possible for you, determining the optimal buying price becomes crucial, making it your decision-making problem. This distinction also illustrates how the buyer's decision-making, as the primary entity, varies based on whether you present your buying price or not.

Ikuta's focus lies on the decision-making of the ***primary entity*** in the Asset Trading Problem, that varies depending on whether or not the primary entity sets the trading price. He distinguishes between two instances: one where the primary entity sets the trading price, termed the "ℙ-mechanism"—a posted price mechanism—and another where the counterpart offers their prices, leaving the primary entity to decide whether to accept it, termed the "ℝ-mechanism"—a reservation price mechanism.

Again, the P-mechanism signifies the decision-making situation where the primary entity must present its trading price to the counterparts, while the ℝ-mechanism represents the situation where the primary entity decides whether to accept or reject the trading price presented by its counterparts.

According to this classification, the Asset Selling Problem (ASP) can be subdivided into two types: one with the ℙ-mechanism, abbreviated as ASP[ℙ], and the other with the ℝ-mechanism, abbreviated as ASP[ℝ]. Similarly, the Asset Buying Problem (ABP) can be subdivided into two types: one with the ℙ-mechanism, abbreviated as ABP[ℙ], and the other with the ℝ-mechanism, abbreviated as ABP[ℝ]. Ikuta refers to these four types—ASP[ℙ], ASP[ℝ], ABP[ℙ], and ABP[ℝ]—of trading problems as the "*Quadruple-asset-trading-problems*."

Ikuta's integrated theory pertains to the relationships among these four types of trading problems. In Section 2, a more detailed exposition of these relationships is provided. Section 3 summarizes the models resulting from the introduction of new elements by Ikuta and outlines the characteristics of the optimal stopping rules within these models. Finally, in Section 4, we touch upon a practical application problem worth considering, enlightened by Ikuta's research.

2. Ikuta's Integrated Theory of Quadruple Asset Trading Problems

Ikuta's integrated theory commences with two fundamental questions: "*Is the* *buying problem always symmetrical to a selling problem?*" and "*Does a general theory integrating the quadruple-asset-trading-problems exist?*"

One might counterpose the first question as follows: Could the essence of the latter problem be comprehended by simply changing the signs of variables, parameters, constants, and similar elements used in the former problem? In reality, many researchers, including myself, have harbored such a perspective that can be described as close to intuition and have, for the most part, not paid substantial attention to it. However, surprisingly, Ikuta's research presents remarkably detailed analytical results contrary to this intuitive viewpoint.

Regarding the second question mentioned above, let's briefly touch upon the background of this inquiry. Ikuta observed optimality equations in numerous research papers related to the asset trading problem within the realm of optimal stopping problems. He gained insight that all these equations are closely connected to a function called the T-function (including its variations, such as the K-function).[1] This observation led him to speculate that there might be a common denominator, making it plausible to construct a general theory integrating the quadruple-asset-trading-problems. Consequently, this insight and prospect bore fruit in the subsequent exploration of the symmetry and analogous relationship within the quadruple-asset-trading-problems.

Refer to Figure 1, a reconstructed simplified version based on figures from Ikuta's work (2023), including Figure 15.1.1, Figure 17.1.4, and Figure 17.1.5.

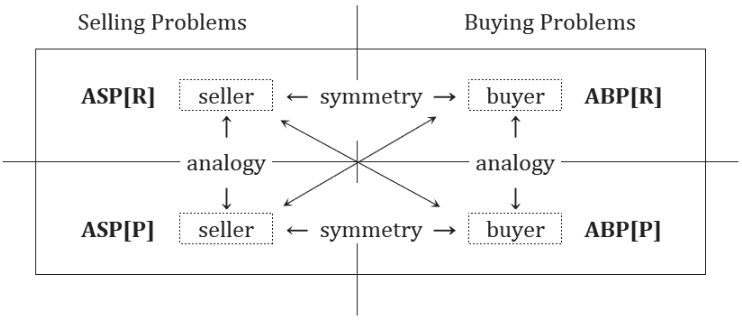

Figure 1: Symmetry and Analogy among the Quadruple-Asset-Trading-Problems

1 $T(x) = \mathbf{E}[\max\{\xi - x, 0\}] = \int_{\infty}^{\infty} \max\{\xi - x, 0\} f(\xi) d\xi$,
$K(x) = \lambda\beta T(x) - (1-\beta)x - s$

This figure illustrates:

〈1〉 Symmetry between ASP[ℝ] and ABP[ℝ], as represented by Ikuta:

$$\mathrm{ASP}\,[\mathbb{R}] \sim \mathrm{ABP}\,[\mathbb{R}]$$

〈2〉 Symmetry between ASP[ℙ] and ABP[ℙ], as represented by Ikuta:

$$\mathrm{ASP}\,[\mathbb{P}] \sim \mathrm{ABP}\,[\mathbb{P}]$$

〈3〉 Analogy between ASP[ℝ] and ASP[ℙ], as represented by Ikuta:

$$\mathrm{ASP}\,[\mathbb{R}] \bowtie \mathrm{ASP}\,[\mathbb{P}]$$

〈4〉 Analogy between ABP[ℝ] and ABP[ℙ], as represented by Ikuta:

$$\mathrm{ABP}\,[\mathbb{R}] \bowtie \mathrm{ABP}\,[\mathbb{P}]$$

〈5〉 Diagonal analogy between ASP[ℝ] and ABP[ℙ], and between ABP[ℝ] and ASP[ℙ], depicted by two-way arrows (↔) on the diagonal.

As evident from this figure, it illustrates the complete cyclic relation among the four types' models in the quadruple-asset-trading-problems. Here, in simple terms, the symmetry relations in both ⟨1⟩ and ⟨2⟩ indicate that, for each of X = ℝ,ℙ, a simple *transformation* of specific variables within ASP[X] corresponds to its inverse ABP[X], and vice versa. Meanwhile, the analogy relations in ⟨3⟩ and ⟨4⟩ signify that *replacing* particular variables within ASP[X] by others results in their corresponding variables within ABP[X], and vice versa.

2.1 Revealing Symmetry and Analogy: A Fresh Perspective on Selling and Buying Problems, and the Collapse of Symmetry and Analogy

Now, let's add some remarks on the *symmetry* relation mentioned in ⟨1⟩ and ⟨2⟩ from the preceding section. To reiterate, in a broad sense, when considering an asset selling problem and its inverse buying problem, one might observe that the characteristics of the latter can be understood by simply altering the signs of variables, constants, and similar elements used in its reverse, the former selling problem. However, this is not a universally applicable law to all buying and selling problems. There are cases where this reversal effect might not hold true, particularly when factors such as information asymmetry between buyers and sellers or market imperfections significantly impact the relationship. Diverging somewhat from the main discussion, Ikuta's interest appears to be focused on the intuitive notion that lies between the selling problem and the buying problem, specifically concentrating on the range of price distributions where this perception holds true.

Ikuta's analysis of the symmetry and analogy in the quadruple-asset-trading-problems is comprehensive and detailed. However, at this point, we can only outline it without delving further. His research indicates that symmetry emerges between the selling problem with the R-mechanism and its inverse buying problem with the R-mechanism (i.e., between ASP[$\mathbb{R}$] and ABP[$\mathbb{R}$]) when expanding the trading price range of ξ to $-\infty<\xi<\infty$, whereas real-world prices are assumed to be positive (i.e., $\xi \in (0,\infty)$). Consequently, he faces a challenge: When restricting the trading price range to $\xi \in (0, \infty)$, does the symmetry observed on $(-\infty, \infty)$ persist on $(0, \infty)$? His study reveals that "it does not persist *very frequently*." Additionally, it unveils a similar pattern between selling and its inverse buying problems with the P-mechanism (i.e., between ASP[$\mathbb{P}$] and ABP[$\mathbb{P}$]). This aspect stands out as a highlight of his findings, revealing the 'collapse of symmetry' —a crucial revelation that we might have overlooked due to what seemed like a simplistic or intuitive viewpoint, including my own.

Interestingly, Ikuta's research reveals that similar observations, as mentioned in the preceding paragraph, can be applied to the *analogy* relations mentioned in ⟨3⟩ and ⟨4⟩ from the preceding section.

These findings hold significance due to the lack of attention in the existing literature, despite its extensive coverage of various applications of optimal stopping problems. This oversight may be attributed to the scarcity of research examples addressing issues involving the P-mechanism. To the best of my knowledge, Peng-Sheng You (1998) stands as the sole example. However, since he did not possess a perspective similar to Ikuta's, he did not address the relationship between the P-mechanism and the R-mechanism.

Ikuta's research also elucidates the diagonal symmetry mentioned in ⟨5⟩. His finding indicates that the symmetry is consistently preserved (or, in other words, the symmetry consistently exists) between the selling problem on $(-\infty, 0)$ and the buying problem on $(0, \infty)$, while the symmetry and analogy mentioned in preceding paragraphs are not always retained on $(0, \infty)$.

2.2 A Brief Summary of Ikuta's Findings and Their Implications

To summarize, two points can be made: (1) Ikuta' work clarifies the entire cyclic relationship among the quadruple models in asset trading problems using the concepts of symmetry and analogy. (2) His work reveals that the previously mentioned symmetry and analogy are not always maintained on $(0, \infty)$, emphasizing their limitations within the real-world price range and indicating the collapse of symmetry and analogy.

Considering that the second point suggests its potential occurrence independently of situations involving the aforementioned 'information asymmetry' between buyers and sellers or 'market imperfections,' his findings offer us new insights and a valuable opportunity to reevaluate the significance of a theoretical finding for a better understanding of the real world. There are various aspects to note about his work and his way of viewing things, giving us a glimpse into his philosophical background. However, for a more in-depth understanding of his work and insightful descriptions about the latter, it is recommended to refer to Ikuta (2023).

3. Ikuta's Innovations in Optimal Stopping Problems: A Comprehensive Exploration of Structured Models

Ikuta's research extends beyond the issues highlighted in the previous section, as he introduces several new elements into optimal stopping problems, elucidating optimal stopping rules in numerous models derived from the quadruple-asset-trading-problems.

3.1 New Elements

In this section, we will focus on the two elements— the quitting penalty and the possibility of search-skipping—among several new elements introduced in Ikuta's research. Additionally, to offer a comprehensive overview of Ikuta's examined models, we will also touch upon the recallability of once-rejected offers.

- Quitting penalty

In the context of optimal stopping problems, the following two fundamental assumptions form the basis for its framework: (1) A seller (or buyer) has to sell (or buy) a specific asset by a deadline, with no allowance for quitting the transaction process without completing the transaction. (2) If the search for counterpart traders is conducted at a point in time, a counterpart appears at the next point in time with a known probability λ ($0<\lambda\leq 1$).

However, when the probability is assumed to be less than 1 (i.e., $0<\lambda<1$), a situation may arise where no counterpart trader appears in the subsequent points in time, even if the search continues. This implies a possible termination of the search process without completing a transaction by the deadline—contradicting the aforementioned first assumption.

To address this issue, Ikuta introduces the element of 'quitting penalty' into his models, which comes in two forms: (1) 'Terminal Quitting Penalty Price,' allowing the primary entity of a selling/buying problem to terminate the trade process at the deadline by paying a penalty price ρ ($\rho>0$); and (2) 'Intermediate Quitting Penalty Price,' allowing the primary entity to halt the trade process at any point in time, besides the deadline.

Consideration of these two types of Quitting Penalty leads to the examination of the following three of kinds trading models:

- **Model 1**, in which the quitting penalty is not available.
- **Model 2,** in which only the terminal quitting penalty is available.
- **Model 3**, in which both the terminal and the intermediate quitting penalty are available.

- Search-skipping possibility

Ikuta introduces another distinctive element in his models, which revolves around whether the search activity for finding a counterpart trader at every point in time is mandatory or discretionary. In the former case, the decision-maker (the primary entity of the transaction) must continue the search until the process is halted by selling/buying a car. Conversely, in the latter case, the decision-maker has the option to either conduct the search or skip it during the decision process (including switching between search to skip). This incorporation adds a layer of realism to Ikuta's models. Ikuta refers to each of these two cases as:

- **Search-Enforced-Case,** or shortly **sE-case,**
- **Search-Allowed-Case,** or shortly **sA-case**.

The combination of the classification based on the availability of the quitting penalty (Model 1, Model 2, Model 3) and the consideration of skipping possibility (sE-case, sA-case) results in a large set of 24 unique types of asset trading models with the R-mechanism or P-mechanism mentioned earlier, as represented by

$$Q\langle \mathrm{M}: x[\mathrm{X}]\rangle = \left\{\mathrm{M}: x[\mathbb{R}][\mathrm{X}], \widetilde{\mathrm{M}}: x[\mathbb{R}][\mathrm{X}], \mathrm{M}: x[\mathbb{P}][\mathrm{X}], \widetilde{\mathrm{M}}: x[\mathbb{P}][\mathrm{X}]\right\},$$
$$x = 1,2,3;\ \mathrm{X} = \mathrm{E}, \mathrm{A}$$

Here, x indicates the number of Model 1, Model 2, and Model 3, while X represents each of the sE-case and sA-case mentioned above. Additionally, M and $\widetilde{\mathrm{M}}$ represent a model for asset selling and buying problem, respectively. The following breakdown will make it easier to comprehend the whole set of 24 types of models:

$$Q\langle \mathrm{M}: 1[\mathrm{E}]\rangle = \left\{\mathrm{M}: 1[\mathbb{R}][\mathrm{E}], \widetilde{\mathrm{M}}: 1[\mathbb{R}][\mathrm{E}], \mathrm{M}: 1[\mathbb{P}][\mathrm{E}], \widetilde{\mathrm{M}}: 1[\mathbb{P}][\mathrm{E}]\right\}$$
$$Q\langle \mathrm{M}: 1[\mathrm{A}]\rangle = \left\{\mathrm{M}: 1[\mathbb{R}][\mathrm{A}], \widetilde{\mathrm{M}}: 1[\mathbb{R}][\mathrm{A}], \mathrm{M}: 1[\mathbb{P}][\mathrm{A}], \widetilde{\mathrm{M}}: 1[\mathbb{P}][\mathrm{A}]\right\}$$
$$Q\langle \mathrm{M}: 2[\mathrm{E}]\rangle = \left\{\mathrm{M}: 2[\mathbb{R}][\mathrm{E}], \widetilde{\mathrm{M}}: 2[\mathbb{R}][\mathrm{E}], \mathrm{M}: 2[\mathbb{P}][\mathrm{E}], \widetilde{\mathrm{M}}: 2[\mathbb{P}][\mathrm{E}]\right\}$$
$$Q\langle \mathrm{M}: 2[\mathrm{A}]\rangle = \left\{\mathrm{M}: 2[\mathbb{R}][\mathrm{A}], \widetilde{\mathrm{M}}: 2[\mathbb{R}][\mathrm{A}], \mathrm{M}: 2[\mathbb{P}][\mathrm{A}], \widetilde{\mathrm{M}}: 2[\mathbb{P}][\mathrm{A}]\right\}$$
$$Q\langle \mathrm{M}: 3[\mathrm{E}]\rangle = \left\{\mathrm{M}: 3[\mathbb{R}][\mathrm{E}], \widetilde{\mathrm{M}}: 3[\mathbb{R}][\mathrm{E}], \mathrm{M}: 3[\mathbb{P}][\mathrm{E}], \widetilde{\mathrm{M}}: 3[\mathbb{P}][\mathrm{E}]\right\}$$
$$Q\langle \mathrm{M}: 3[\mathrm{A}]\rangle = \left\{\mathrm{M}: 3[\mathbb{R}][\mathrm{A}], \widetilde{\mathrm{M}}: 3[\mathbb{R}][\mathrm{A}], \mathrm{M}: 3[\mathbb{P}][\mathrm{A}], \widetilde{\mathrm{M}}: 3[\mathbb{P}][\mathrm{A}]\right\}$$

- Recallability of once-rejected offer

The recallability of a once-rejected offer plays a crucial role in the decision-making process. If a model assumes that a once-rejected offer at a previous point in time cannot be recalled later and accepted, it is referred to as a *model-without-recall*. Conversely, a model in which such recallability is available is referred to as a *model-with-recall*.

Ikuta, therefore, proposes and examines the optimality equations to find optimal decision rules for each of the 24 unique types of asset trading models-without-recall and the other 24 models-with-recall, resulting in a total of 48 models. Discussing the optimality equations of each model as a whole is beyond the scope of this brief review. Therefore, for detailed information, refer to Ikuta (2023).

3.2 Four Kinds of Optimal Decisions and the Most Highlighted Finding

The key questions for decision-makers in the aforementioned models examined by Ikuta can be categorized into the following four:

〈1〉 Whether to accept or reject the price proposed by a counterpart (relevant for models with R-mechanism)

〈2〉 What price to post (relevant for models with P-mechanism),

〈3〉 Whether to conduct or skip the search (relevant for models in the sA-case), 〈4〉 When to initiate the process, i.e., the search activity (applicable to all models).

The fourth decision-making aspect, among the above-mentioned four categories, plays a predominant role in Ikuta's research, as it is applicable to all of his models. We can consider that the emergence of this decision is attributed to the introduction of the element 'Search-skipping possibility' into all models. Simultaneously, the necessity to delve into the *optimal timing to initiate* the search arises, encompassing all points in time within its scope. Let's add a more detailed explanation for this point.

In conventional models of optimal stopping problems, it is assumed that a decision-maker (e.g., a seller) *starts* the search for potential counterparts (buyers) at the 'starting point' in time and *stops* the decision process by accepting an offer presented by a buyer up to the deadline. In other words, the focus is on determining the optimal '*stopping time,*' which falls between the starting time and the deadline. However, Ikuta introduces a novel approach by establishing more tailored time points based on the human decision-making process: the "*recognizing time*", "*starting time*", "*initiating time*", "*stopping time*", and "*deadline.*"

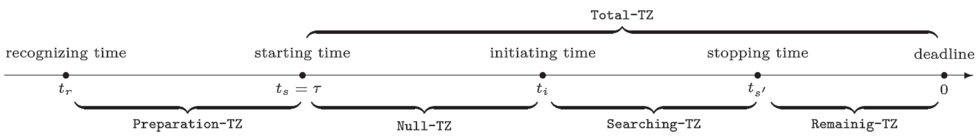

Figure 2: Five Time-zones

Look at Figure 2. To simplify the explanation of the five points in time based on Ikuta's definitions, the decision problem materializes when the decision-maker (e.g., a seller) first *recognizes* it (the 'recognizing time'). However, it takes some time to embark on the problem-solving process. After a preparatory period, the decision-maker then reaches the time point to *start* the problem-solving process (the 'starting time').

At this point, the decision-maker can either immediately *initiate* searching for potential counterparts (e.g., buyers) or strategically consider initiating a more opportune searching time (the 'initiating time'). In the case of the former, the 'starting time' and the 'initiating time' overlap; in the case of the latter, a temporal gap occurs between the starting time and the initiating time, represented as the 'Null-Time-Zone' referred to by Ikuta in Figure 2 (Source: Ikuta (2023)).

During this period, the decision-maker assesses the optimal time to initiate a well-considered search, taking into account expected values from initiating the search on the next day or the day after tomorrow, and so on, compared to a later initiation. Following the determined initiating time, the decision-maker is compelled to decide whether to accept or reject the emerging offers until accepting a counterpart's offer and stopping the decision-making process (the 'stopping time'). [2]

In this context, Ikuta's focus is concentrated on analyzing and identifying the *optimal initiating time*. Through extensive and meticulous analyses, he provides evidence that the optimal initiating time can be proven to fall into one of the following three categories, especially for the optimal stopping models with no-recall, each based on specific conditions:

A) It is optimal to initiate the process at the starting time,
B) It is optimal to initiate the process at the deadline,
C) It is optimal to initiate the process between the starting time and the deadline.

The first finding is interesting, but the second one is even more intriguing. The second finding suggests that when the null-time-zone extends across all points in time on the planning horizon except the deadline, it eventually follows that all decision-making activities throughout the entire planning horizon becomes futile, except for the deadline. Ikuta metaphorically likens this situation to "as if all matters, even light, falling into a black hole," and refers to it as "deadline falling"—this could be considered the most prominent discovery of his study.

4. Concluding Remarks

The following provides a brief overview of Ikuta's study, highlighting its distinctive features through the following summaries:

- Objective:
  - Establishing an integrated theory of the asset trading problems based on the concepts of symmetry and analogy.
- New research approaches:

2 For further explanation involving the search-skipping possibility, I leave it to the readers' interpretation.

- Presenting asset trading models with the R-mechanism and with the P-mechanism.
- Introducing the Quitting-penalty and the Search-skipping-possibility
- Introducing Starting-time and Initiating-time.
- Extending research scope to the 48 asset trading models.

• Key Findings:

- Symmetry and analogy, including diagonal symmetry, between the Quadruple-asset-trading-problems.
- Discovery of the Null-time zone and Deadline-falling.
- Definition of the 48 unique asset trading models including many new models that have not been discussed in the existing literature.
- Optimal decision rules based on the optimal-initiating time for each model.
- Definition and introduction of new functions for formulating optimal stopping problems, such as the K-function, including other variants of L- and $\mathcal{L}$-functions (although the latter two functions are not mentioned in this review).

It may be appropriate to state that Ikuta's research represents a significant departure from existing literature on optimal stopping problems. In particular, his research prompts the introduction of the Quitting-penalty and the Search-skipping-possibility into my previous studies, for example, Kang (2005). To my best knowledge, Ee and Ikuta (2006) is the first research that introduced the two elements, the Quitting-penalty and the Search-skipping-possibility, into the field of optimal stopping problems.

While Ikuta's approach provides numerous avenues for new research, one intriguing application could be exploring how the optimal exercise timing for American call and put options aligns with Ikuta's perspective on optimal initiating time; in other words, how it is affected when introducing Ikuta's optimal initiating time concept.

Regarding the research on optimal exercise timing for an American call option, Tabata and Sawaki (1989, 1984) have previously examined the optimal exercise policies. However, several challenges remain unresolved. The aim of exploring the aforementioned application is to confirm the optimal exercise policies already examined in the existing literature and to potentially discover new properties associated with the optimal exercise policies. A potential avenue for this extension could involve a collaborative work between Ikuta and myself.

Finally, I would like to make one clarification. The scope of Ikuta's research is so extensive that it cannot be fully encapsulated in this review, which remains a brief overview. To gain a thorough understanding of his work, it is inevitable to delve into Ikuta's original manuscripts. Hopefully, this short review will be somewhat helpful in reading and understanding the extensive volumes of Ikuta's research. However, the depth and breadth of his research can only be truly appreciated by reading Ikuta's own writings.

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