Iteratively Forcing Fast Functions

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Abstract

This paper presents a simple example of iterated forcing that utilizes the newly developed Aspero-Mota method. The Aspero-Mota method facilitates iterative forcing with a class of partially ordered sets (posets) such that a size of the continuum gets strictly greater than the second uncountable cardinal. The class of posets encompassed by this method is larger than the class of countable chain condition (ccc) posets. This larger class of posets includes the one that forces a generic fast function from a generic closed cofinal subset of the least uncountable cardinal into the least uncountable cardinal. The new Aspero-Mota method makes use of many transitive set universes that satisfy fragments of set theory and their countable elementary substructures. It introduces a “marker” to control how a condition functions with respect to the relevant elementary substructures. Here, a different rendition of this method is presented by simply iterating the fast function poset and using only one transitive set universe that satisfies a fragment of set theory. The markers are interpreted as initial segmets of the single universe’s relevant elementary substructures.

Introduction

Aspero-Mota introduces a new method of iteratively forcing a class of V-finitely proper posets, where V denotes the ground model, in [AM]. Roughly speaking, to iterate proper posets without collapsing cardinals, the elementary substructures of transitive set universes that satisfy fragments of set theory are necessary. In the case of V-finitely proper posets, a morass-like family of elementary substructures is needed, which requires a condition that is simultaneously generic for all of them. V-finitely proper posets provide such a condition. [AM] uses markers and many transitive set universes $H_\theta = \{ x \mid \text{the size of the transitive closure of } x \text{ is of a size strictly less than } \theta\}$, where θs are regular uncountable cardinals, that satisfy fragments of set theory. The Aspero-Mota method is presented here by simply iterating a typical non-ccc poset. The fast function poset is iteratively forced. As in [M], a second-order treatment of iterated forcing $\langle P_\alpha \mid \alpha \leq \kappa \rangle$ is provided as a subset of H_κ, where κ is a regular cardinal with $\kappa \geq \omega_2$. The P_αs have the ω_2-cc under the Continuum Hypothesis (CH). This treatment of iterated forcing makes sense when $P_\alpha \subseteq H_\kappa$ and P_α have the κ-cc.

§ 0. Preliminary

Let us fix a regular cardinal κ with $\kappa \geq \omega_2$ for the rest of this paper. Let $\langle P_\alpha \mid \alpha \leq \kappa \rangle$ be an iterated forcing such that $P_\alpha \subseteq H_\kappa$ and has the κ-cc for all $\alpha \leq \kappa$.

As in [M], we prepare basic facts. For the sake of concise presentation, we employ abbreviations. Though we omit to write, P_α has associated objects such as a partial order, a greatest element, a set of P-names $V^P \cap H_\kappa$ and a forcing relation for equality $\{(p, \tau, \pi) \mid p \Vdash_{P_\alpha} \text{“}\tau = \pi\text{”}\} \cap H_\kappa$. Any sequence $\langle S_\beta \mid \beta < \alpha \rangle$ of subsets of H_κ is coded as a subset $\langle\langle S_\beta \mid \beta < \alpha \rangle\rangle = \{(\beta, s) \mid \beta < \alpha,\ s \in S_\beta\}$ of H_κ. We write $N \prec (H_\kappa, \cdots)$ to mean that N is a countable elementary substructure of a relational structure $(H_\kappa, \cdots)$.

Lemma. Let $\rho < \alpha \leq \kappa$. For any formula $\varphi(x_1, \cdots, x_n)$, there exists a formula $\varphi^*(x_1, \cdots, x_n, y, z)$ such that for any $p \in P_\alpha$ and any $\tau_1, \cdots, \tau_n \in H_\kappa \cap V^{P_\rho}$,

$$p \Vdash_{P_\rho} \text{“}(H_\kappa^{V[\dot{G}_\rho]}, \in, \mathrm{H}_\kappa^V, P_\rho, G_\rho, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha \rangle\rangle \models \text{“}\varphi(\tau_1, \cdots, \tau_n)\text{””}.$$

iff

$$(H_\kappa, \in, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha \rangle\rangle \models \text{“}\varphi^*(\tau_1, \cdots, \tau_n, \rho, p)\text{”}.$$

In particular,

$$1 \Vdash_{P_\rho} \text{“}(H_\kappa^{V[\dot{G}_\rho]}, \in, H_\kappa^V, P_\rho, \mathrm{G}_\rho, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha \rangle\rangle \models \text{“}\exists y\, \varphi(y, \tau_1, \cdots, \tau_n)\text{””}.$$

iff

$$(H_\kappa, \in, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha \rangle\rangle) \models \text{“}\exists\, y : P_\rho\text{–name s.t. } \varphi^*(y, \tau_1, \cdots, \tau_n, \rho, 1)\text{”}.$$

Proof. We point out two important items. The rests are routine checking.

- Since P_αs are subset of H_κ and have the κ-cc, $H_\kappa^{V[G_\rho]} = \{\pi_{G_\rho} \mid \pi \in H_\kappa \cap V^{P_\rho}\}$.
- For $p, x \in P_\rho$, $p \Vdash_{P_\rho}$“$x \in \dot{G}_\rho$” iff for any $q \leq p$ in P_ρ, there exists $r \in P_\rho$ such that $r \leq q, x$ in P_ρ.

□

Lemma. Let $\alpha \leq \kappa$. Let $\mathrm{N} \prec (H_\kappa, \in, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha \rangle\rangle)$.

(1) Let G_α be P_α-generic over V. Then

$$N[G_\alpha] \prec (H_\kappa^{V[G_\alpha]}, \in, H_\kappa^V, P_\alpha, G_\alpha, \langle\langle P_\beta \mid \beta < \alpha \rangle\rangle).$$

(2) Let $\rho \in N \cap \alpha$. Then $N \prec (H_\kappa, \in, P_\rho, \langle\langle P_\beta \mid \beta < \rho \rangle\rangle)$.
(3) Let $\rho \in N \cap \alpha$. Let G_ρ be P_ρ-generic over V. Then

$$N[G_\rho] \prec (H_\kappa^{V[G_\rho]}, \in, H_\kappa^V, P_\rho, G_\rho, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha \rangle\rangle).$$

Proof. For (3): Let $\tau_1, \cdots, \tau_n \in N \cap V^{P_\rho}$. Since

$$1 \Vdash_{P_\rho} \text{“}(H_\kappa^{V[\dot{G}_\rho]}, \in, H_\kappa^V, P_\rho, G_\rho, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha\rangle\rangle) \models \text{“}\exists\, x\ \forall y\ (\varphi(y, \tau_1, \cdots, \tau_n) \Longrightarrow \varphi(x, \tau_1, \cdots, \tau_n))\text{””},$$

and

$$\tau_1, \cdots, \tau_n, \rho, 1 \in N \prec (H_\kappa, \in, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha\rangle\rangle),$$

there exists $\pi \in N \cap V^{P_\rho}$ such that

$$1 \Vdash_{P_\rho} \text{“}(H_\kappa^{V[\dot{G}_\rho]}, \in, H_\kappa^V, P_\rho, G_\rho, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha\rangle\rangle) \models \text{“}\forall y\ (\varphi(y, \tau_1, \cdots, \tau_n) \Longrightarrow \varphi(\pi, \tau_1, \cdots, \tau_n))\text{””}.$$

Hence, for any P_ρ-generic G_ρ over V, if

$$(H_\kappa^{V[G_\rho]}, \in, H_\kappa^V, P_\rho, G_\rho, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha\rangle\rangle) \models \text{“}\exists y\, \varphi(y, (\tau_1)_{G_\rho}, \cdots, (\tau_n)_{G_\rho})\text{”},$$

then

$$(H_\kappa^{V[G_\rho]}, \in, H_\kappa^V, P_\rho, G_\rho, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha\rangle\rangle) \models \text{“}\varphi((\pi)_{G_\rho}, (\tau_1)_{G_\rho}, \cdots, (\tau_n)_{G_\rho})\text{”}.$$

Notation. We write $N \prec \mathcal{P}_{\leq\alpha}$ to indicate that N is a countable elementary substructure of the relational structure $\mathcal{P}_{\leq\alpha} = (H_\kappa, \in, P_\alpha, \langle\langle P_\beta \mid \beta < \alpha\rangle\rangle)$. In particular, if $N \prec \mathcal{P}_{\leq\alpha}$ and $\alpha < \kappa$, then $\alpha \in N \cap \kappa$ holds. If $N \prec \mathcal{P}_{\leq\alpha}$ and $\beta \in N \cap \alpha$, then $N \prec \mathcal{P}_{\leq\beta}$ holds. If $N \prec \mathcal{P}_{\leq\alpha}$ and G_α is P_α-generic over V, then $N[G_\alpha]$ is an elementary substructure of an expanded relational structure $(H_\kappa^{V[G_\alpha]}, \in, H_\kappa^V, P_\alpha, G_\alpha, \langle\langle P_\beta \mid \beta < \alpha\rangle\rangle)$ in the generic extension $V[G_\alpha]$. In particular, H_κ^V, P_α, and G_α are available as unary predicates.

Predense subsets are used to formulate generic conditions in this paper.

Definition. Let P be a poset such that $P \subseteq H_\kappa$ and P has the κ-cc. Let N be a countable elementary substructure of a relational structure $(H_\kappa, \in, P)$. We say $q \in P$ is (P,N)-generic, if for any predense subset D of P with $D \in N$, $D \cap N$ is predense below q.

Lemma. Let $q \in P$ and $N \prec (H_\kappa, \in, P)$ be as above. The following are equivalent.
- q is (P,N)-generic.
- $p \Vdash_P$ “$N[\dot{G}] \cap H_\kappa^V = N$”.
- $p \Vdash_P$ “$N[\dot{G}] \cap \kappa = N \cap \kappa$”

Here, $N[\dot{G}] = \{\tau_{\dot{G}} \mid \tau \in N \cap V^P\}$.

If $P \in H_\kappa$, then $P \in N \prec (H_\kappa, \in)$ iff $N \prec (H_\kappa, \in, P)$. In this case, $q \in P$ is (P,N)-generic iff for any dense subset $D \in N$ of P, $D \cap N$ is predense below q.

§ 1. The Fast Function Poset

We explicate the fast function poset. We specifically deal with a partial function from ω_1 to ω_1.

Definition. Let $p \in P$, if p is a finite partial function from ω_1 into ω_1 such that if $i, j \in \mathrm{dom}(p)$ with $i < j$, then $i \leq p(i) < j$. Let $p, q \in P$, then $q \leq p$ in P, if $q \supseteq p$.

We call this forcing poset, the fast function (forcing) poset. The fast function poset does not have the ccc. However, it has nice properties. In particular, it is a proper poset.

Proposition. (1) Let $p \in P$ and $p, P \in N$, where N is a countable elementary substructure of H_κ. Then $p \cup \{(N \cap \omega_1, N \cap \omega_1)\}$ is (P,N)-generic.
(2) Let G be P-generic over the ground model V. Let $\dot{f} = \cup G$. Then $\dot{f}$ is a partial function from ω_1 into ω_1 such that the domain of $\dot{f}$ is a closed cofinal subset of ω_1.
(3) For any function f from ω_1 to ω_1 with $f \in V$, $\{x < \omega_1 \mid f(x) < \dot{f}(x)\}$ is uncountable.

Proof. We provide some details.

For (1): Let $p \in P$ and N be a countable elementary substructure of H_κ such that $p, P \in N$. Let $D \in N$ be a predense subset of P. We want to show that $D \cap N$ is predense below $p \cup \{(N \cap \omega_1, N \cap \omega_1)\}$. Let (q, d) be such that $q \leq p \cup \{(N \cap \omega_1, N \cap \omega_1)\}$, $q \leq d$, and $d \in D$. It suffices to find (h^+, d') such that $h^+ \leq q, d'$ and $d' \in D \cap N$. Since

$$H_\kappa \models \text{“There exists } (q', d') \text{ s.t. } q' \in P, d' \in D, q' \leq d', \text{ and } q' \leq q \cap N\text{”},$$

$$P, D, q \cap N \in N \prec H_\kappa,$$

there exists $(q', d') \in N$ as such. Let $h^+ = q' \cup q$. Then $h^+ \in P$ such that $h^+ \leq q, q'$, and $q' \leq d' \in D \cap N$.

For (2): We show that $\mathrm{dom}(\dot{f})$ is closed. Let $\xi < \omega_1$ be a limit ordinal such that $\mathrm{dom}(\dot{f}) \cap \xi$ is cofinal below ξ. We want to show that $\xi \in \mathrm{dom}(\dot{f})$. To the contrary, suppose $\xi \notin \mathrm{dom}(\dot{f})$. Let $p \in G$ such that $p \Vdash_P$ “$\mathrm{dom}(\dot{f}) \cap \xi$ is cofinal below ξ and $\xi \in \mathrm{dom}(\dot{f})$”. We may assume that $\mathrm{dom}(p) \setminus \xi \neq \emptyset$. Let η_1 be the $<$-least member of $\mathrm{dom}(p) \setminus \xi$. Let η_0 be the $<$-greatest member of $\mathrm{dom}(p) \cap \xi$. Then $\eta_0 \leq p(\eta_0) < \xi < \eta_1$. Let $q = p \cup \{(p(\eta_0)+1, \xi)\}$. Then $q \in P$, $q \leq p$, and $q \Vdash_P$ “$\mathrm{dom}(\dot{f}) \cap \xi$ has the $<$-greatest member $p(\eta_0)+1 < \xi$”.
This would be a contradiction.

□

Note that for any function $\dot{g}$ from ω_1 to ω_1 in any ccc generic extension of V, there exists a function $f \in V$ from ω_1 to ω_1 such that for all $x < \omega_1$, $\dot{g}(x) < f(x)$. This contrasts the ccc posets and the fast function poset.

§ 2. Iteration

We recursively construct a sequence of posets $\langle P_\alpha \mid \alpha \leq \kappa \rangle$. We first try to give an intuition behind the formal definition. Any condition $p = (\mathcal{N}^p, S^p, A^p) = (\mathcal{N}, S, A) \in P_\alpha$ consists of three parts. The first part $\mathcal{N}$ is a finite set of well-organized countable elementary subtructures of $(H_\kappa, \cdots)$. The $\mathcal{N}$ contributes to establish the chain conditions and properness of P_α. Second part is a relation S from $\mathcal{N}$ to α. The S tells for each $N \in \mathcal{N}$, the coordinates $\xi < \alpha$ where the iterand $A(\xi)$ is $N[\dot{G}_\xi]$-generic witnessed by $p \lceil \xi$. We demand $S(N) = \{\xi < \alpha \mid NS\xi\}$ is an initial segment of $N \cap \alpha$. The relation S is usually an infinite set but essentially of finite by this initial segment requirement. The third part A is the woking part of p. The A lists a finite set of decided values of finite fragments of generic fast functions at the finitely many coordinates dom(A). We represent A as a finite relation from α to $\omega_1 \times \omega_1$. Hence $A(\xi) = \{(i, j) \mid \xi A(i, j)\}$ are all in the fast function poset, all but finitely many are the empty sets. If $NS\xi$, then $p \lceil \xi$ witnesses that the fast function forced at ξ is closed below $N \cap \omega_1$. Hence this $N \in \mathcal{N}$ prevents p to collapse ω_1 by the generic fast function at ξ . We now begin a formal presentation.

Definition. Let $\alpha \leq \kappa$. Let p = $(\mathcal{N}^p, S^p, A^p) \in P_\alpha$, if

(ob):
- $\mathcal{N}^p$ is a finite symmetric system of countable elementary substructures of $(H_\kappa, \in)$. Namely,
  - If $N, M \in \mathcal{N}^p$ with $N =_{\omega_1} M$, then $(N, \in, \mathcal{N}^p \cap N)$ and $(M, \in, \mathcal{N}^p \cap M)$ are isomorphic such that the unique isomorphism is the identity on the intersection $N \cap M$, where $N =_{\omega_1} M$ abbreviates $N \cap \omega_1 = M \cap \omega_1$.
  - If $N, M \in \mathcal{N}^p$ with $N <_{\omega_1} M$, then there exists $M' \in \mathcal{N}^p$ such that $N \in M'$ and $M' =_{\omega_1} M$, where $N <_{\omega_1} M$ abbreviates $N \cap \omega_1 < M \cap \omega_1$.
- S^p is a relation from $\mathcal{N}^p$ to α such that for all Y, $S^p(Y) = \{\eta \mid Y S^p \eta\}$ is an initial segment of $Y \cap \alpha$.
- A^p is a finite relation from α to $\omega_1 \times \omega_1$ such that for all $\xi < \alpha$, $A^p(\xi) = \{(i, j) \mid (\xi, (i, j)) \in A^p\}$ is a finite fast function from ω_1 to ω_1.
  (el): If $Y S^p \eta$, then $Y \prec \mathcal{P}_{\leq \eta}$.
  (g): If $Y S^p \eta$ and $A^p(\eta) \neq \emptyset$, then $Y \cap \omega_1 \in$ dom $(A^p(\eta))$.
  For $p, q \in P_\alpha$, $q \leq p$ in P_α, if

$$\mathcal{N}^q \supseteq \mathcal{N}^p, S^q \supseteq S^p, \text{ and } A^q \supseteq A^p.$$

We note that a marker γ of $N \in \mathcal{N}^p$ is $\gamma \leq \alpha$ such that $S^p(N) = N \cap \gamma$, denoted like $(N, \gamma) \in \Delta^p$ in [AM], though we do not make use of this non-unique ordinal in this paper. We first observe that P_αs form an iterated forcing in the following two senses. They in turn explicitely tell how to calculate P_ρ-generic filters over V from P_α-generic filters over V, where $\rho < \alpha$. The proofs are routine checkings and left to the readers.

Lemma. (Projection) Let $\rho < \alpha \leq \kappa$.

(1) If $p \in P_\alpha$, then $p \lceil \rho = (\mathcal{N}^p \lceil \rho, S^p \lceil \rho, A^p \lceil \rho) \in P_\rho$, where

$$\mathcal{N}^p \lceil \rho = \mathcal{N}^p,$$

$$S^p \lceil \rho = \{(Y, \eta) \mid \eta < \rho, Y S^p \eta\},$$
$$A^p \lceil \rho = \{(\xi, (x, y)) \mid \xi < \rho, \xi A^p (x, y)\}.$$

(2) For $p, q \in P_\alpha$, if $q \leq p$ in P_α, then $q \lceil \rho \leq p \lceil \rho$ in P_ρ.
(3) For $p \in P_\alpha$ and $h \in P_\rho$ with $h \leq p \lceil \rho$, let

$$h^+ = (\mathcal{N}^h \cup \mathcal{N}^p, S^h \cup S^p, A^h \cup A^p).$$

Then $h^+ \in P_\alpha$ such that $h^+ \lceil \rho = h$ and $h^+ \leq p$ in P_α.

Lemma. (Complete Embedding) Let $\rho < \alpha \leq \kappa$.

(1) $P_\rho \subset P_\alpha$.
(2) For $p, q \in P_\rho$, $q \leq p$ in P_ρ iff in P_α.
(3) For $p, q \in P_\rho$, p, q are compatible in P_ρ iff in P_α.
(4) For $p \in P_\alpha$, $p \leq p\lceil\rho$ in P_α.

By the lemmas above, we have an explicit calculation of generic objects in the generic extension $V[G_\alpha]$.

Lemma. Let $\rho < \alpha \leq \kappa$. Let G_α be P_α-generic over V. Let $G_\alpha\lceil\rho = \{p\lceil\rho \mid p \in G_\alpha\}$. Then $G_\alpha\lceil\rho$ is P_ρ-generic over V. And we have

$$G_\alpha\lceil\rho = G_\alpha \cap P_\rho.$$

We observe that P_αs have the ω_2-cc assuming CH. In particular, the cardinals $\geq \omega_2$ remain to be cardinals.

Lemma. (CH) $P_\alpha \subset H_\kappa$ and has the ω_2-cc.

Proof. Let $\langle p_i \mid i < \omega_2 \rangle$ be an indexed family of conditions of P_α. Then we may pick elementary substructures $\langle N_i \mid i < \omega_2 \rangle$ of H_θ, where θ is a sufficiently large regular cardinal, such that $p_i, P_\alpha \in N_i$. By CH, we may thin these N_is and may assume that N_is form a Δ-system, N_is are all isomorphic and the isomorphisms are the identities on the intersections $N_i \cap N_j$. Let $q = (\mathcal{N}^{p_i} \cup \mathcal{N}^{p_j}, S^{p_i} \cup S^{q_j}, A^{p_i} \cup A^{p_j})$. Then $q \in P_\alpha$ and $q \leq p_i, p_j$.

□

The next two lemmas combined assure that any condition may be extended to a generic condition. In particular, ω_1 remains to be ω_1.

Lemma. Let $p \in P_\alpha$ and $p \in X \prec \mathcal{P}_{\leq\alpha}$. Let

$$q = (\mathcal{N}^p \cup \{X\}, S^p \cup \{(X, \eta) \mid \eta \in X \cap \alpha\}, A^p \cup \{(\xi, (X \cap \omega_1, X \cap \omega_1)) \mid A^p(\xi) \neq \emptyset\}.)$$

Then $q \in P_\alpha$, $q \leq p$ in P_α, and $S^q(X) = X \cap \alpha$ (the largest possible).

Here is the main lemma that shows P_αs are all proper.

Lemma. Let $p \in P_\alpha$, $S^p(X) = X \cap \alpha$ (the largest possible), and $X \prec \mathcal{P}_{\leq\alpha}$, then p is (P_α, X)-generic.

Proof. We simply write the structure $(H_\kappa^{V[G_\alpha]}, \in, H_\kappa^V, P_\alpha, G_\alpha, \cdots)$ by $(H_\kappa^{V[G_\alpha]}, \cdots)$. We know that if $N \prec \mathcal{P}_{\leq\alpha}$, then $N[G_\alpha] \prec (H_\kappa^{V[G_\alpha]}, \cdots)$, where G_α is P_α-generic over V and $N[G_\alpha] = \{\tau_{G_\alpha} \mid \tau \in V^{P_\alpha} \cap N\}$. ([M])

Case 1. successor, let, $\alpha = \alpha + 1$: Let $p \in P_{\alpha+1}$, $S^p(X) = X \cap (\alpha + 1)$, and $X \prec \mathcal{P}_{\leq\alpha+1}$. Let $D \in X$ be predense in $P_{\alpha+1}$. Let $q \leq p$, d in $P_{\alpha+1}$ and $d \in D$. We may assume that $A^q(\alpha) \neq \emptyset$. Hence the value $A^q(\alpha)(X \cap \omega_1)$ gets defined. Since $q\lceil\alpha \in P_\alpha$, $S^{q\lceil\alpha}(X) = X \cap \alpha$, and $X \prec \mathcal{P}_{\leq\alpha}$, by induction $q\lceil\alpha$ is (P_α, X)-generic. Let G_α be P_α-generic over V with $q\lceil\alpha \in G_\alpha$. We argue in $V[G_\alpha]$. Since

$$(H_\kappa^{V[G_\alpha]}, \cdots) \models \text{“}\exists (q', d') \text{ s.t. } q' : P_{\alpha+1}, q' \lceil \alpha : G_\alpha, q' \leq d', A^q(\alpha) \cap X \subset A^{q'}(\alpha) \text{ and } d' \in D\text{”},$$

and

$$\alpha, A^q(\alpha) \cap X, D \in X[G_\alpha] \prec (H_\kappa^{V[G_\alpha]}, \in, \cdots),$$

we have $h \in P_\alpha$ and $(q', d') \in X$ such that

- $q' \in P_{\alpha+1}$, $d' \in D$ such that $A^p(\alpha) \cap X \subset A^{q'}(\alpha)$, $q' \leq d'$ in $P_{\alpha+1}$,
- $h \leq q' \lceil \alpha, q \lceil \alpha$ in P_α.

Let

$$\mathcal{N}^{h^+} = \mathcal{N}^h \cup \mathcal{N}^{q'} \cup \mathcal{N}^q = \mathcal{N}^h,$$
$$S^{h^+} = S^h \cup S^{q'} \cup S^q = S^h \cup \{(Y, \alpha) \mid Y S^{q'} \alpha\} \cup \{(Y, \alpha) \mid Y S^q \alpha\},$$
$$A^{h^+} = A^h \cup A^q \cup A^{q'} = A^h \cup (\{\alpha\} \times (A^{q'}(\alpha) \cup A^q(\alpha))).$$

Then $h^+ \in P_{\alpha+1}$, $h^+ \lceil \alpha = h$, and $h^+ \leq q, q'$. Hence $h^+ \leq q, d' \in D \cap X$.

□

Case 2. $\mathrm{cf}(\alpha) = \omega$: Let $p \in P_\alpha$, $S^p(X) = X \cap \alpha$, and $X \prec \mathcal{P}_{\leq \alpha}$. Let $D \in X$ be predense in P_α. Let $q \leq p$, d such that $d \in D$. Choose $\rho \in X \cap \alpha$ such that

- $\mathrm{dom}(A^q) \subset \rho$.

Let us consider $q \lceil \rho$. Then $q \lceil \rho \in P_\rho$, $S^{q \lceil \alpha}(X) = X \cap \rho$, and $X \prec \mathcal{P}_{\leq \rho}$. By induction, $q \lceil \rho$ is (P_ρ, X)-generic. Let G_ρ be P_ρ-generic over V with $q \lceil \rho \in G_\rho$. We argue in $V[G_\rho]$. Since

$$(H_\kappa^{V[G_\rho]}, \cdots) \models \text{“}\exists (q', d') \text{ s.t. } q' : P_\alpha, q' \lceil \rho : G_\rho, d' \in D, \mathrm{dom}(A^{q'}) \subset \rho, \text{ and } q' \leq d'\text{”},$$

and

$$\rho, D \in X[G_\rho] \prec (H_\kappa^{V[G_\rho]}, \cdots),$$

we have $h \in P_\rho$ and $(q', d') \in X$ such that

- $q' \in P_\alpha$, $d' \in D$, $\mathrm{dom}(A^{q'}) \subset \rho$, and $q' \leq d'$ in P_α,
- $h \leq q' \lceil \rho, q \lceil \rho$ in P_ρ.

Let

$$\mathcal{N}^{h^+} = \mathcal{N}^h \cup \mathcal{N}^{q'} \cup \mathcal{N}^q = \mathcal{N}^h,$$
$$S^{h^+} = S^h \cup S^{q'} \cup S^q = S^h \cup \{(Y, \eta) \mid Y S^{q'} \eta \geq \rho\} \cup \{(Y, \eta) \mid Y S^q \eta \geq \rho\},$$
$$A^{h^+} = A^h \cup A^{q'} \cup A^q = A^h \cup \emptyset \cup \emptyset.$$

Then $h^+ \in P_\alpha$, $h^+ \lceil \rho = h$, and $h^+ \leq q', q$. Hence $h^+ \leq q, d' \in D \cap X$.

□

Case 3. $\mathrm{cf}(\alpha) > \omega$: Let $p \in P_\alpha$, $S^p(X) = X \cap \alpha$, and $X \prec \mathcal{P}_{\leq \alpha}$. Let us write $\alpha_X = \sup(X \cap \alpha)$. Let

$D \in X$ be predense in P_α. Let $q \leq p, d$ with $d \in D$. Let us choose $\rho \in X \cap \alpha$ such that

- $\mathrm{dom}(A^q) \cap \alpha_X \subset \rho$,
- For any $Y \in \mathcal{N}^q$, if $Y <_{\omega_1} X$, then $Y \cap X \cap \alpha \subset \rho$.

Let us consider $q \lceil \rho$. Then $q \lceil \rho \in P_\rho$, $S^{q \lceil \rho}(X) = X \cap \rho$, and $X \prec \mathcal{P}_{\leq \rho}$. By induction, $q \lceil \rho$ is (P_ρ, X)-generic. Let G_ρ be P_ρ-generic over V with $q \lceil \rho \in G_\rho$. We argue in $V[G_\rho]$. Since

$$(H_\kappa^{V[G_\rho]}, \cdots) \models \text{“}\exists (q', d') \text{ s.t. } q' : P_\alpha, q' \lceil \rho \in G_\rho, d' \in D, q' \leq d'\text{”}$$

and

$$\rho, D \in X[G_\rho] \prec (H_\kappa^{V[G_\rho]}, \cdots),$$

we have $h \in P_\rho$ and $(q', d') \in X$ such that

- $q' \in P_\alpha$, $d' \in D$, and $q' \leq d'$,
- $h \leq q' \lceil \rho, q \lceil \rho$.

Let

$$\mathcal{N}^{h^+} = \mathcal{N}^h \cup \mathcal{N}^{q'} \cup \mathcal{N}^q = \mathcal{N}^h,$$
$$S^{h^+} = S^h \cup S^{q'} \cup S^q = S^h \cup \{(Y, \eta) \mid Y S^{q'} \eta \geq \rho\} \cup \{(Y, \eta) \mid Y S^q \eta \geq \rho\}.$$
$$A^{h^+} = A^h \cup A^{q'} \cup \{(\eta, (Y \cap \omega_1, Y \cap \omega_1)) \mid \eta \in [\rho, \alpha_X), A^{q'}(\eta) \neq \emptyset, X \leq_{\omega_1} Y, Y S^q \eta\} \cup A^q.$$

Then $h^+ \in P_\alpha$, $h^+ \lceil \rho = h$, and $h^+ \leq q', q$ in P_α. Hence $h^+ \leq q, d' \in D \cap X$.
We check (g) for h^+: Let $Y S^{h^+} \eta$ and $A^{h^+}(\eta) \neq \emptyset$.

Case. $\eta < \rho$: Then $Y S^h \eta$ and $A^h(\eta) \neq \emptyset$. Hence the value $A^h(\eta)(Y \cap \omega_1)$ defined.
Case. $\rho \leq \eta < \alpha_X$: Then $Y S^{q'} \eta$ or $Y S^q \eta$, and $A^{q'}(\eta) \neq \emptyset$.
Subcase. $Y S^{q'} \eta$ and $A^{q'}(\eta) \neq \emptyset$: Then the value $A^{q'}(\eta)(Y \cap \omega_1)$ defined.
Subcase. $Y S^q \eta$ and $A^{q'}(\eta) \neq \emptyset$: Then $\rho \leq \eta \in Y \cap X \cap \alpha$. Hence $X \leq_{\omega_1} Y$. By definition,

$$A^{h^+}(\eta)(Y \cap \omega_1) = Y \cap \omega_1.$$

Case. $\alpha_X \leq \eta < \alpha$: $Y S^q \eta$ and $A^q(\eta) \neq \emptyset$: Then the value $A^q(\eta)(Y \cap \omega_1)$ defined.

□

We conclude this paper with the following that is impossible to show by the ccc posets.

Theorem. (CH) Let G_κ be P_κ-generic over V. Then in the generic extension $V[G_\kappa]$, we have a family $\langle f_\alpha \mid \alpha < \kappa \rangle$ such that

(1) f_α is a partial function from a closed cofinal subset C_α of ω_1 to ω_1 such that for all $x \in C_\alpha$, $x \leq f_\alpha(x)$.
(2) For any function f from ω_1 to ω_1, there exists $\alpha < \kappa$ such that for all $\beta \in [\alpha, \kappa)$, $\{x < \omega_1 \mid f(x) < f_\beta(x)\}$ is uncountable.

Proof. Let $\dot{f}_\alpha = \cup \{A^p(\alpha) \mid p \in G_\kappa\}$. We claim that the $\dot{f}_\alpha$s work. We observe (2). Let $\dot{f}$ be a P_κ-

name such that $\Vdash_{P_\kappa}$"$\dot{f}: \omega_1 \longrightarrow \omega_1$". By the ω_2-cc, we may assume that there exists $\alpha < \kappa$ such that $\dot{f}$ is a P_α-name. Let $\alpha \leq \beta < \kappa$, p $\in P_{\beta+1}$ and $t < \omega_1$. It suffices to show that there exists $q \in P_{\beta+1}$ and $t < x < \omega_1$ such that $q \leq p$ and $q \Vdash_{P_{\beta+1}}$"$\dot{f}(x) < \dot{f}_\beta(x)$". To this end, let $t < x < \omega_1$ with $A^p(\beta) \subset x$. Let $p' \leq p \lceil \beta$ be such that there exists y with $p' \Vdash_{P_\beta}$"$\dot{f}(x) = y$". Let $q = (\mathcal{N}^{p'}, S^{p'} \cup S^p, A^{p'} \cup A^p \cup \{(\beta, (x, \max\{x, y\}+1))\})$. Then $q \in P_{\beta+1}$, $q \leq p$, and $q \Vdash_{P_{\beta+1}}$"$\dot{f}(x) = y < y + 1 \leq A^q(\beta)(x) = \dot{f}_\beta(x)$".

□

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