Forcing a Club by a Generalized Fast Function

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Abstract

We present a new proper poset that includes the least uncountable cardinal's closed and cofinal generic subset D. A generic uncountable partial function S is forced by the poset. We call S a generalized fast function. The domain of S is the set of D-accumulation points plus the smallest element of D. For any consecutive two elements $x < y$ in the domain of S, the function S maps x to S_x in such a way that S_x is a subset of the open interval (x,y), is order-isomorphic to the set of the natural numbers and is cofinal below y. The elements of D in (x,y) are provided by S_x. The poset consists of finite conditions. In particular, S_x are forced by their initial segments.

Then, we present a proper poset that forces an uncountable generic family of countable models of a set theory fragment. This family was dubbed a morass-like symmetric system after it was projected down to a simplified gap-1 morass of Velleman. This post combines the poset's machinery for the generalized fast function with that of Aspero and Mota. As a result, the poset is made up of finite conditions and has a relevant chain condition.

Introduction

In [M2], we discuss a proper poset P composed of finite conditions and force an uncountable generic partial function f from ω_1 into ω_1. Let D represent the domain of f. D is then a closed and cofinal subset of ω_1. Let us use c_D to transitively collase D onto ω_1. Let zero$(D) = \{d \in D \mid c_D(d) = 0\}$, suc$(D) = \{d \in D \mid c_D(d)$ is a successor ordinal$\}$, and lim$(D) = \{d \in D \mid c_D(d)$ is a limit ordinal$\}$. Then, they are the singleton set $\{0\}$, a set of successor ordinals, and a set of limit ordinals. In fact, for any $d \in D$, the next element of d in D is $f(d)+1$. Hence, D does not have a flexible partition.

In [M1], we find connections between Aspero and Mota's ([AM]) symmetric systems and morass-like structures, including simplified $(\omega_1,1)$-morasses of Velleman ([V1]). In general, we want an uncountable generic family $\mathcal{N}$ of countable models of a set theory fragment, where $\mathcal{N}$ is symmetric as in [AM] and the projection $D = \{\omega_1 \cap N \mid N \in \mathcal{N}\}$ is a closed and cofinal subset of ω_1. Furthermore, we want $\mathcal{N}$ to be partitioned into the three cells zero$(\mathcal{N})$, suc$(\mathcal{N})$, and lim$(\mathcal{N})$ such that zero$(D) = \{\omega_1 \cap N \mid N \in$ zero $(\mathcal{N})\}$, suc$(D) = \{\omega_1 \cap N \mid N \in$ suc $(\mathcal{N})\}$, and lim$(D) = \{\omega_1 \cap N \mid N \in$ lim$(\mathcal{N})\}$. Then, the projection $\mathcal{A} = \{\omega_2 \cap N \mid N \in \mathcal{N}\}$ is a simplified $(\omega_1,1)$-morass as in [V1], although it is not as neat.

We first present a new proper poset P_{GFF} that forces C, $\langle S_\delta \mid \delta \in C \rangle$, and D such that (1) C

consists of countable limit ordinals and is a closed and cofinal subset of ω_1. (2) For any consecutive two elements $\delta < \delta^+$ of C, S_δ is a cofinal subset in the open interval (δ, δ^+) and the order-type of S_δ is ω. (3) P_{GFF} forces C by the finite fragments and the S_δ by the associated initial segments. To form D, we take the union of C and the S_δ (all $\delta \in C$). As a result, each condition $p \in P_{\mathrm{GFF}}$ knows which elements in its relevant field are in the cell suc(D) and in the union of the two cells zero$(D) \cup$ lim(D). We prefer the formulation in which zero(D) is dependent on the generic filter.

Then, we present a new poset P_{MOR}, which is a variation on posets in [AM] and incorporates P_{GFF} machinery. Let H_{ω_2} denote the set of sets x whose transitive closures have sizes less than ω_2. Then, H_{ω_2} is a transitive set model of a set theory fragment. The variant P_{MOR} forces a generic family of countable elementary H_{ω_2}-substructures. The family is known as a morass-like symmetric system, and it descends to an $(\omega_1,1)$-morass of Velleman ([V1]).

What is left to the readers is combining variations of these posets iteratively to force, say, a two-sorted morass-like symmetric system that carves out an $(\omega_1,2)$-morass of Velleman ([V2]).

§ 1. Generalized Fast Functions

We force a generalized fast function.

Definition. Let $p = f^p \in P_{\mathrm{GFF}}$, if f^p is a function such that

- dom(f^p) is a finite set of countable limit ordinals.
- For each $\delta \in$ dom(f^p), $f^p(\delta)$ is a finite set of countable ordinals x of any kind with $\delta < x$.
- For each $\delta_1 < \delta_2$ in dom(f^p) and each $x \in f^p(\delta_1)$, we demand $x < \delta_2$.

For p, q in P_{GFF}, let $q \leq p$ in P_{GFF}, if dom$(f^q) \supseteq$ dom(f^p) and for each $\delta \in$ dom(f^p), $f^q(\delta)$ end-extends $f^p(\delta)$, i.e., $f^q(\delta) \supseteq f^p(\delta)$ and for any $x \in f^p(\delta)$ and $y \in f^q(\delta)$, if $y < x$, then $y \in f^p(\delta)$.

Here are easy densities.

Lemma. (1) For any $p \in P_{\mathrm{GFF}}$ and any $\eta < \omega_1$, there is $q \leq p$ in P_{GFF} such that there exists $\delta \in$ dom(f^q) with $\eta < \delta$.
(2) For any $p \in P_{\mathrm{GFF}}$, any consecutive two elements $\delta < \delta^+$ in dom(f^p), and any $\eta < \delta^+$, there exists $q \leq p$ in P_{GFF} s.t. $f^q(\delta)$ is a proper end-extension of $f^p(\delta)$ and the largest element of $f^q(\delta)$ is greater than η.

We observe that P_{GFF} is proper.

Lemma. Let λ be a sufficiently large regular cardinal with $P_{\mathrm{GFF}} \in H_\lambda$. Let M be a countable elementary substructure of H_λ with $P_{\mathrm{GFF}} \in M$. Let $q \in P$ with $\omega_1 \cap M \in$ dom(f^q). Then q is (P_{GFF}, M)-generic.

Proof. Let us simply denote P for P_{GFF}. Let D be a predense subset of P with $D \in M$. We want to show that $D \cap M$ is predense below q in P. We provide a typical construction that would be dubbed an amalgamation. By performing amalgamations densely below q in P, we would be done. Thus here simply suppose that there exists $d \in D$ with $q \leq d$ in P. We first fix M-copies (q', d', δ') of $(q, d, \omega_1 \cap M)$ as follows. Since

$M \prec H_\lambda \models$ "$\exists (q', d', \delta')$ s.t. $q' \in P$, $d' \in D$, $\delta' \in \mathrm{dom}(f^{q'})$, $q' \leq d'$ in P, and $f^q \restriction (\omega_1 \cap M) = f^{q'} \restriction \delta'$",

and relevant parameters, say, P, D, $f^q \restriction (\omega_1 \cap M)$ are all in M, we have $(q', d', \delta') \in M$ as described. Now we form a common extension r of q and q' in P, where $r = q' \cup q$. Notice that $r \leq d' \in D \cap M$.

□

Lemma. Let G be P_{GFF}-generic over the ground model V. In the generic extension $V[G]$, we form $\dot{C}$, $\dot{S}_\delta$ (for each $\delta \in \dot{C}$), and $\dot{D}$ as follows.

$$\dot{C} = \bigcup \{\mathrm{dom}(f^p) \mid p \in G\},$$

$$\dot{S}_\delta = \bigcup \{f^p(\delta) \mid p \in G,\ \delta \in \mathrm{dom}(f^p)\},$$

$$\dot{D} = \dot{C} \cup \bigcup \{\dot{S}_\delta \mid \delta \in \dot{C}\}.$$

Then

(1) $\dot{C}$ consists of countable limit ordinals and is a closed and cofinal subset of ω_1.

(2) For any consecutive two elements $\delta < \delta^+$ of $\dot{C}$, $\dot{S}_\delta$ is a subset of the open interval (δ, δ^+), is cofinal below δ^+, and is of order-type ω.

(3) $\dot{D}$ is a closed and cofinal subset of ω_1 s.t.
- The least element of $\dot{D}$ is the least element c_0 of $\dot{C}$.
- The set of accumulation points of $\dot{D}$ is $\dot{C} \setminus \{c_0\}$.
- For any consecutive two elements $\delta < \delta^+$ in $\dot{C}$, we have $\dot{S}_\delta = (\delta, \delta^+) \cap \dot{D}$.

  We may call the map $\langle \delta \mapsto \dot{S}_\delta \mid \delta \in \dot{C} \rangle$ a generalized fast function.

Proof. We just outline that $\dot{C}$ is closed. Suppose η is a countable limit ordinal such that $\eta \cap \dot{C}$ is cofinal below η witnessed by $w \in G$. We may assume that there exist (δ_0, δ_1) s.t. δ_0 is the largest element of $\mathrm{dom}(f^w) \cap \eta$ and δ_1 is the least element of $\{\delta \in \mathrm{dom}(f^w) \mid \eta \leq \delta\}$. Argue by contradiction that $\eta = \delta_1$. Hence $\eta \in \dot{C}$ witnessed by w itself.

□

§ 2. Morass-like Symmetric Systems

We introduce a proper poset P_{MOR} that incorporates the machinery of P_{GFF} and forces a type of symmetric system $\dot{\mathcal{N}}$ of [AM]. Due to the machinery, the projection $\{\omega_1 \cap N \mid N \in \dot{\mathcal{N}}\}$ is a closed and cofinal subset of ω_1 and the projection $\{\omega_2 \cap N \mid N \in \dot{\mathcal{N}}\}$ is an $(\omega_1,1)$-morass of Velleman in [V1], though it is not a neat one yet. We set notations C and $\lim(C)$. Let

$$C = \{N \in [H_{\omega_2}]^\omega \mid N \prec H_{\omega_2}\}.$$

$$\lim(C) = \{M \in C \mid M = \bigcup (C \cap M)\}.$$

Hence for any $M \in \lim(C)$ and any $x \in M$, there exists $N \in C$ with $x \in N \in M$. Since N is countable and $M \prec H_{\omega_2}$ knows about it, it entails that N is a proper subset of M.

Definition. We say a finite subfamily $\mathcal{N}$ of C is morass-like symmetric (a morass-like symmetric system), if $\mathcal{N}$ satisfies the following.

- (iso) For any $N, M \in \mathcal{N}$, if $\omega_1 \cap N = \omega_1 \cap M$, then there exists an $\in$-isomorphism $\phi_{NM}: N \longrightarrow M$ s.t. $\phi_{NM}(x) = x$ for all $x \in N \cap M$.
- (up) For any $N_3, N_2 \in \mathcal{N}$, if $\omega_1 \cap N_3 < \omega_1 \cap N_2$, then there exists $N_1 \in \mathcal{N}$ s.t. $N_3 \in N_1$ and $\omega_1 \cap N_1 = \omega_1 \cap N_2$.
- (down) For any $N_1, N_2, N_3 \in \mathcal{N}$, if $N_3 \in N_1$ and $\omega_1 \cap N_1 = \omega_1 \cap N_2$, then $\phi_{N1N2}(N_3) \in \mathcal{N}$.
- For each $N \in \mathcal{N}$, either (Zero), (One), or (Two) holds, where
  (Zero) $\mathcal{N} \cap N = \emptyset$.
  (One) There exists $N_0 \in \mathcal{N} \cap N$ s.t. $\mathcal{N} \cup N = \{N_0\} \cap (\mathcal{N} \cap N_0)$.
  (Two) There exist $N_1, N_2 \in \mathcal{N} \cap N$ s.t.
  $\star\, \omega_1 \cap N_1 = \omega_1 \cap N_2$,
  $\star\, \Delta := (\omega_2 \cap N_1) \cap (\omega_2 \cap N_2) < (\omega_2 \cap N_1) \setminus \Delta < (\omega_2 \cap N_2) \setminus \Delta \neq \emptyset$,
  $\star\, \mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_2) \cup (\mathcal{N} \cap N_1)$.
  Notice if $\mathcal{N}$ satisfies (iso), (up), and (down), then $\mathcal{N}$ satisfies (sliding) and (interpolating) by [AM].
  (sliding) For each $N_3, N_2 \in \mathcal{N}$, if $\omega_1 \cap N_3 < \omega_1 \cap N_2$, then there exists $N_4 \in \mathcal{N} \cap N_2$ with $\omega_1 \cap N_4 = \omega_1 \cap N_3$.
  (interpolating) For each $N_3, N, N_1 \in \mathcal{N}$, if $\omega_1 \cap N_3 < \omega_1 \cap N < \omega_1 \cap N_1$ and $N_3 \in N_1$, then there exists $M \in \mathcal{N}$ s.t. $N_3 \in M \in N_1$ and $\omega_1 \cap M = \omega_1 \cap N$.

Next suppose we are in a cardinal preserving generic extension $V[G]$ over the ground model V. We say an infinite subfamily $\dot{\mathcal{N}} \in V[G]$ of C is morass-like symmetric (a morass-like symmetric system), if $\dot{\mathcal{N}}$ satisfies (iso), (up), (down), (par), and (cof), where

- (iso) For any $N, M \in \dot{\mathcal{N}}$, if $\omega_1 \cap N = \omega_1 \cap M$, then there exists an $\in$-isomorphism $\phi_{NM}: N \longrightarrow M$ s.t. $\phi_{NM}(x) = x$ for all $x \in N \cap M$.
- (up) For any $N_3, N_2 \in \dot{\mathcal{N}}$, if $\omega_1 \cap N_3 < \omega_1 \cap N_2$, then there exists $N_1 \in \dot{\mathcal{N}}$ s.t. $N_3 \in N_1$ and $\omega_1 \cap N_1 = \omega_1 \cap N_2$.
- (down) For any $N_1, N_2, N_3 \in \dot{\mathcal{N}}$, if $N_3 \in N_1$ and $\omega_1 \cap N_1 = \omega_1 \cap N_2$, then $\phi_{N1N2}(N_3) \in \dot{\mathcal{N}}$.
- (par) $\dot{\mathcal{N}}$ gets partitioned into the three cells $\mathrm{zero}(\dot{\mathcal{N}})$, $\mathrm{suc}(\dot{\mathcal{N}})$, and $\lim(\dot{\mathcal{N}})$, where

$$\mathrm{zero}(\dot{\mathcal{N}}) = \{N \in \dot{\mathcal{N}} \mid \dot{\mathcal{N}} \cap N = \emptyset\},$$
$$\mathrm{suc}(\dot{\mathcal{N}}) = \{N \in \dot{\mathcal{N}} \mid \text{either (One) or (Two)}\}, \text{ where}$$

(One) There exists $N_0 \in \dot{\mathcal{N}} \cap N$ s.t. $\dot{\mathcal{N}} \cap N = \{N_0\} \cap (\dot{\mathcal{N}} \cap N_0)$.
(Two) There exist $N_1, N_2 \in \dot{\mathcal{N}} \cap N$ s.t.
$\star\, \omega_1 \cap N_1 = \omega_1 \cap N_2$,
$\star\, \Delta := (\omega_2 \cap N_1) \cap (\omega_2 \cap N_2) < (\omega_2 \cap N_1) \setminus \Delta < (\omega_2 \cap N_2) \setminus \Delta \neq \emptyset$,

$\star \dot{\mathcal{N}} \cap N = \{N_1, N_2\} \cup (\dot{\mathcal{N}} \cap N_2) \cup (\dot{\mathcal{N}} \cap N_1)$.

$$\lim(\dot{\mathcal{N}}) = \{N \in \dot{\mathcal{N}} \mid N = \bigcup(\dot{\mathcal{N}} \cap N)\}.$$

- (cof) $\bigcup \dot{\mathcal{N}} = (H_{\omega_2})^V$.
  Notice that if $\dot{\mathcal{N}}$ is (cof), then it is (dir), where

- (dir) For any $N, M \in \dot{\mathcal{N}}$, there exists $K \in \dot{\mathcal{N}}$ with $N, M \in K$.

Notice that for any morass-like symmetric system, it is necessary that the projection $\{\omega_1 \cap N \mid N \in \dot{\mathcal{N}}\}$ is a closed and cofinal subset of ω_1. We force a stationary morass-like symmetric system.

Definition. Let $p = (f^p, \mathcal{N}^p) \in P_{\mathrm{MOR}}$, if
- $f^p \in P_{\mathrm{GFF}}$.
- $\mathcal{N}^p$ is a finite morass-like symmetric subfamily of C s.t. $\bigcup \mathcal{N}^p \in \mathcal{N}^p$.
- $\mathrm{dom}(f^p) \cup \bigcup \{f^p(\delta) \mid \delta \in \mathrm{dom}(f^p)\} = \{\omega_1 \cap N \mid N \in \mathcal{N}^p\}$.
- For each $N \in \mathcal{N}^p$, if $\omega_1 \cap N \in \mathrm{dom}(f^p)$, then we demand not just $N \in C$ but $N \in \lim(C)$.
  For $p, q \in P$, $q \leq p$ in P, if $f^q \leq f^p$ in P_{GFF} and $\mathcal{N}^q \supseteq \mathcal{N}^p$.

Lemma. (Freeze) For any $p \in P_{\mathrm{MOR}}$, any $\delta \in \mathrm{dom}(f^p)$, any consecutive two $\eta < \eta^+$ in $\{\delta\} \cup f^p(\delta)$, and any $N \in \mathcal{N}^p$, if $\omega_1 \cap N = \eta^+$ and N has a pair $N_1, N_2 \in \mathcal{N}^p \cap N$ that satisfy (Two), then for any $q \leq p$ in P_{MOR}, the pair remains so with respect to q. In particular, $\mathcal{N}^q \cap N = \{N_1, N_2\} \cap (\mathcal{N}^q \cap N_1) \cup (\mathcal{N}^q \cap N_2)$ holds.

Proof. The consecutive two $\eta < \eta^+$ in $\{\delta\} \cup f^p(\delta)$ remain so in $\{\delta\} \cup f^q(\delta)$. Since $N \in \mathcal{N}^q$ and $\omega_1 \cap N = \eta^+$, either (One) or (Two) holds with (q, N, η, η^+). But $N_1, N_2 \in \mathcal{N}^q \cap N$ and $\omega_1 \cap N_1 = \omega_1 \cap N_2 = \eta$. Hence $\{M \in \mathcal{N}^p \cap N \mid \omega_1 \cap M = \eta\} = \{N_1, N_2\} = \{M \in \mathcal{N}^q \cap N \mid \omega_1 \cap M = \eta\}$.

$\square$

Lemma. (Dense) (1) For any $p \in P_{\mathrm{MOR}}$ and any $\eta < \omega_1$, there exists $q \leq p$ in P_{MOR} s.t. there exists $\delta \in \mathrm{dom}(f^q)$ with $\eta < \delta$.
(2) For any $p \in P_{\mathrm{MOR}}$, any two consecutive $\delta < \delta^+$ in $\mathrm{dom}(f^p)$, and any $M \in \mathcal{N}^p$ with $\omega_1 \cap M = \delta^+$, and any $x \in M$, there exists (q, N) s.t. $q \leq p$ in P_{MOR} and $N \in \mathcal{N}^q$ with $x \in N$.

Proof. For (1), we use the assumption $\bigcup \mathcal{N}^p \in \mathcal{N}^p$ to extend. We outline (2). Since $M \in \lim(C)$, we have $M = \bigcup(C \cap M)$. Since $x, f^p(\delta), \mathcal{N}^p \cap M \in M$, there exists $N \in C \cap M$ s.t. $x, f^p(\delta), \mathcal{N}^p \cap M \in N$. Let $q = (f^q, \mathcal{N}^q)$, where f^q and f^p are the same except $f^q(\delta) = f^p(\delta) \cup \{\omega_1 \cap N\}$ and $\mathcal{N}^q$ is formed by considering appropriate copies of N as follows.

$$\mathcal{N}^q = \{\phi_{MM'}(N) \mid M' \in \mathcal{N}^p, \omega_1 \cap M' = \omega_1 \cap M\} \cup \mathcal{N}^p.$$

Then this (q, N) works.

Lemma. (Proper) Let λ be a sufficiently large regular cardinal. Let M be a countable elementary substructure of H_λ with, say, C, $P_{\mathrm{MOR}} \in M$. In particular, we have $H_{\omega_2} \cap M \in \lim(C)$. Let $q \in P_{\mathrm{MOR}}$ s.t. $H_{\omega_2} \cap M \in \mathcal{N}^q$ and $\omega_1 \cap M \in \mathrm{dom}(f^q)$. Then q is (P_{MOR}, M)-generic.

Proof. We first observe that $H_{\omega_2} \cap M = \bigcup(C \cap (H_{\omega_2} \cap M))$. Let $x \in H_{\omega_2} \cap M$. It suffices to find $N \in C \cap M = C \cap (H_{\omega_2} \cap M)$ with $x \in N$. Since

$$C, x \in M \prec H_\lambda \models \text{“}\exists\ N \in C\ \ x \in N\text{”},$$

we have $N \in C \cap M$ with $x \in N$.

Let us simply denote P for P_{MOR}. Let D be predense in P with $D \in M$. We want to show $D \cap M$ is predense below q. We present a typical argument assuming that there exists $d \in D$ with $q \leq d$ in P. It suffices to argue similarly dense below q. We first fix M-copies (q', d', M') of (q, d, M_1), where $M_1 = H_{\omega_2} \cap M$, as follows. Since

$$H_\lambda \models \text{“}\exists\ (q', d', M')\text{ s.t. } q' \leq d' \text{ in } P,\ d' \in D,\ M' \in \mathcal{N}^{q'},\ (f^q \lceil M_1) = f^{q'} \lceil M',\ (\mathcal{N}^q \cap M_1) = \mathcal{N}^{q'} \cap M'\text{”}$$

and

$$P, D, f^q \lceil M_1, \mathcal{N}^q \cap M_1 \in M \prec H_\lambda,$$

we have $(q', d', M') \in M$ as described. We then form a commom extension r of q' and q in P_{MOR} as follows. Let $f^r = f^{q'} \cup f^q$ and consider appropriate copies of $\mathcal{N}^{q'}$ to form the least finite symmetric

$$\mathcal{N}^r = \{\phi_{M_1 M_2}(K) \mid K \in \mathcal{N}^{q'}, M_2 \in \mathcal{N}^q,\ \omega_1 \cap M_2 = \omega_1 \cap M_1\} \cup \mathcal{N}^q.$$

Note that if $K \in \lim(C) \cap M_1$, then $\phi_{M_1 M_2}(K) \in \lim(C)$. Notice that $r \leq d' \in D \cap M$.

To form copies, we also have

Lemma. Let λ be a sufficiently large regular cardinal. Let M_1, M_2, M_3 be three countable elementary substructures of H_λ s.t. C, $P_{\mathrm{MOR}} \in M_1 \cap M_2$, $\{M_1, M_2\} \in M_3$, and there exists an isomorphism $\phi : M_1 \longrightarrow M_2$ that satisfies $\phi(x) = x$ for all $x \in M_1 \cap M_2$. Furthermore,

$$\Delta := (\omega_2 \cap M_1) \cap (\omega_2 \cap M_2) < (\omega_2 \cap M_1) \setminus \Delta < (\omega_2 \cap M_2) \setminus \Delta \neq \emptyset.$$

Let $p \in P_{\mathrm{MOR}} \cap M_1$ and let $p' = \phi(p)$. Then $p' = (f^p, \phi(\mathcal{N}^p)) \in P_{\mathrm{MOR}} \cap M_2$ and p and p' have a common extension r in P_{MOR} s.t. $H_{\omega_2} \cap M_1, H_{\omega_2} \cap M_2 \in \mathcal{N}^r$ and $\bigcup \mathcal{N}^r = H_{\omega_2} \cap M_3$.

Proof. Routine.

□

Lemma. (CH) P_{MOR} has the ω_2-cc.

Proof. Let $\langle p_i \mid i < \omega_2 \rangle$ be an indexed family of conditions of P_{MOR}. We want to find $i < j$ s.t. p_i and p_j have a common extension r in P_{MOR}. To this end, let λ be a sufficiently large regular cardinal. For each $i < \omega_2$, let us fix a countable elementary substructure M_i of H_λ with C, P_{MOR}, $p_i \in M_i$. By CH, we may assume that $\langle M_i \mid i < \omega_2 \rangle$ forms a Δ-system s.t. for any $i < j < \omega_2$, M_i

and M_j are isomorphic by the map $\phi_{ij}: M_i \longrightarrow M_j$ s.t. $\phi_{ij}(p_i) = p_j$ and $\phi_{ij}(x) = x$ for all $x \in M_i \cap M_j$. Furthermore,

$$\Delta := (\omega_2 \cap M_i) \cap (\omega_2 \cap M_j) < (\omega_2 \cap M_i) \setminus \Delta < (\omega_2 \cap M_j) \setminus \Delta \neq \emptyset.$$

Fix any two $i < j < \omega_2$. Then p_i and p_j are compatible in P_{MOR}.

Lemma. Let G be P_{MOR}-generic over the ground model V. In the generic extension $V[G]$, we form $\dot{C}$, $\dot{S}_\delta$ (for each $\delta \in \dot{C}$), $\dot{D}$, and $\dot{\mathcal{N}}$ as follows.

$$\dot{C} = \bigcup\{\mathrm{dom}(f^p) \mid p \in G\},$$

$$\dot{S}_\delta = \bigcup\{f^p(\delta) \mid \delta \in \mathrm{dom}(f^p), p \in G\},$$

$$\dot{D} = \dot{C} \cup \bigcup\{\dot{S}_\delta \mid \delta \in \dot{C}\},$$

$$\dot{\mathcal{N}} = \bigcup\{\mathcal{N}^p \mid p \in G\}.$$

Then

- $\dot{C}$ is a closed and cofinal subset of ω_1.
- For any two consecutive $\delta < \delta^+$ elements of $\dot{C}$, $\dot{S}_\delta$ is a subset of the open interval (δ, δ^+), cofinal below δ^+, and is of order-type ω.
- $\dot{D}$ is a closed and cofinal subset of ω_1 s.t. the least element of $\dot{D}$ is c_0, where c_0 is the least element of $\dot{C}$, the set of accumulation points of $\dot{D}$ is $\dot{C} \setminus \{c_0\}$, and the set of successor points of $\dot{D}$ is $\bigcup\{\dot{S}_\delta \mid \delta \in \dot{C}\}$.
- $\dot{D} = \{\omega_1 \cap N \mid N \in \dot{\mathcal{N}}\}$.
- $\mathrm{zero}(\dot{\mathcal{N}}) = \{N \in \dot{\mathcal{N}} \mid \omega_1 \cap N$ is the least element of $\dot{D}\}$.
- $\mathrm{suc}(\dot{\mathcal{N}}) = \{N \in \dot{\mathcal{N}} \mid \omega_1 \cap N$ is a successor point of $\dot{D}\}$.
- $\lim(\dot{\mathcal{N}}) = \{N \in \dot{\mathcal{N}} \mid \omega_1 \cap N$ is an accumulation point of $\dot{D}\}$.
- $\dot{\mathcal{N}}$ gets partitioned into the three cells $\mathrm{zero}(\dot{\mathcal{N}})$, $\mathrm{suc}(\dot{\mathcal{N}})$, and $\lim(\dot{\mathcal{N}})$.
- $\dot{\mathcal{N}}$ is morass-like symmetric.

Proof. We outline on $\mathrm{zero}(\dot{\mathcal{N}})$, $\mathrm{suc}(\dot{\mathcal{N}})$, and $\lim(\dot{\mathcal{N}})$.

- $\mathrm{zero}(\dot{\mathcal{N}}) = \{N \in \dot{\mathcal{N}} \mid \omega_1 \cap N = c_0\}$.

Proof. $\subseteq$: Let $N \in \mathrm{zero}(\dot{\mathcal{N}})$. Then $\dot{\mathcal{N}} \cap N = \emptyset$. We know that $\omega_1 \cap N \in \dot{D}$. Suppose $\omega_1 \cap N > c_0$. By (up) and (down), or simply by (sliding), there must be $K \in \dot{\mathcal{N}}$ s.t. $K \in \dot{\mathcal{N}} \cap N$. This is a contradiction.

$\supseteq$: We know c_0 is the least element of $\dot{D}$. Hence $\dot{\mathcal{N}} \cap N = \emptyset$.

□

- $\mathrm{suc}(\dot{\mathcal{N}}) = \{N \in \dot{\mathcal{N}} \mid \exists \delta \in \dot{C}\ \omega_1 \cap N \in \dot{S}_\delta\}$.

Proof. $\subseteq$: Let $N \in \mathrm{suc}(\dot{\mathcal{N}})$. Then

$$\omega_1 \cap N \in \dot{D} = \dot{C} \cup \bigcup\{\dot{S}_\delta \mid \delta \in \dot{C}\} = \{\omega_1 \cap M \mid M \in \dot{\mathcal{N}}\}.$$

Let $\omega_1 \cap N \in \dot{C}$. Since $\dot{\mathcal{N}} \cap N \neq \emptyset$, we have $\omega_1 \cap N > c_0$. We know the set of accumulation points $\lim(\dot{D})$ of $\dot{D}$ is $\dot{C} \setminus \{c_0\}$. Hence $\omega_1 \cap N \in \lim(\dot{D})$. Hence there exist no $N_1, N_2 \in \dot{\mathcal{N}} \cap N$ s.t. $\omega_1 \cap N_1 = \omega_1 \cap N_2$ and

$$\dot{\mathcal{N}} \cap N = \{N_1, N_2\} \cup (\dot{\mathcal{N}} \cap N_1) \cup (\dot{\mathcal{N}} \cap N_2).$$

$\supseteq$: Let $N \in \dot{\mathcal{N}}$ and $\delta \in \dot{C}$ s.t $\omega_1 \cap N \in \dot{S}_\delta$. Let $\eta < \eta^+$ be two consecutive members of $\{\delta\} \cup \dot{S}_\delta$ with $\eta^+ = \omega_1 \cap N$. We have two cases.

Case. There exists a unique $N_0 \in \dot{\mathcal{N}} \cap N$ s.t. $\omega_1 \cap N_0 = \eta$. Then $\dot{\mathcal{N}} \cap N = \{N_0\} \cup (\dot{\mathcal{N}} \cap N_0)$.

Case. There exist $N_1, N_2 \in \dot{\mathcal{N}} \cap N$ s.t. $\omega_1 \cap N_1 = \omega_1 \cap N_2 = \eta$ and

$$\Delta := (\omega_2 \cap N_1) \cap (\omega_2 \cap N_2) < (\omega_2 \cap N_1) \setminus \Delta < (\omega_2 \cap N_2) \setminus \Delta \neq \emptyset.$$

Then

$$\dot{\mathcal{N}} \cap N = \{N_1, N_2\} \cup (\dot{\mathcal{N}} \cap N_1) \cup (\dot{\mathcal{N}} \cap N_2).$$

Hence $N \in \mathrm{suc}(\dot{\mathcal{N}})$.

$\square$

- $\lim(\dot{\mathcal{N}}) = \{N \in \dot{\mathcal{N}} \mid \omega_1 \cap N \in \lim(\dot{D})\}$.
  Proof. $\subseteq$: Let $N \in \lim(\dot{\mathcal{N}})$. Since $\bigcup(\dot{\mathcal{N}} \cap N) = N$, we have $\omega_1 \cap N \in \lim(\dot{D})$. We know $\lim(\dot{D}) = \dot{C} \setminus \{c_0\}$.

$\supseteq$: Let $N \in \dot{\mathcal{N}}$ with $\omega_1 \cap N \in \lim(\dot{D})$. We have two cases.

Case. $(\omega_1 \cap N) \cap \dot{C}$ is cofinal below $\omega_1 \cap N$. We know

$$\omega_1 \cap N \in \dot{C} = \bigcup\{\mathrm{dom}(f^p) \mid p \in G\}.$$

Hence for any $x \in N$, there exists $M \in \dot{\mathcal{N}} \cap N$ with $x \in M$. Hence

$$N = \bigcup(\dot{\mathcal{N}} \cap N).$$

Case. $(\omega_1 \cap N) \cap \dot{C}$ is bounded below $\omega_1 \cap N$. Since $\omega_1 \cap N \in \lim(\dot{D}) = \dot{C} \setminus \{c_0\}$, we have $\delta < \delta^+$ s.t. δ is the largest element of $(\omega_1 \cap N) \cap \dot{C}$, $\delta < \delta^+$ are two consecutive elements of $\dot{C}$ s.t. $\omega_1 \cap N = \delta^+$. Then $\dot{S}_\delta$ is cofinal below $\omega_1 \cap N$. For any $x \in N$, there exists $M \in \dot{\mathcal{N}} \cap N$ with $x \in M$. Hence

$$N = \bigcup(\dot{\mathcal{N}} \cap N).$$

$\square$

Lemma. For $N_3, N_2 \in \dot{\mathcal{N}}$, the following (1) and (2) are equivalent.

(1) $\omega_2 \cap N_3$ is a proper subset of $\omega_2 \cap N_2$.
(2) There exists $N_4 \in \dot{\mathcal{N}} \cap N_2$ s.t. $\omega_2 \cap N_3 = \omega_2 \cap N_4$.

Proof. (1) implies (2) : By taking intersections, we have $\omega_1 \cap N_3 \leq \omega_1 \cap N_2$. If $\omega_1 \cap N_3 =$

$\omega_1 \cap N_2$, then by (iso), we have $\phi_{N_3N_2}[\omega_2 \cap N_3] = \omega_2 \cap N_2$. But $\phi_{N_3N_2}$ fixes $N_3 \cap N_2$ pointwise. Hence $\phi_{N_3N_2}[\omega_2 \cap N_3] = \omega_2 \cap N_3$. Then $\omega_2 \cap N_2 = \omega_2 \cap N_3$. This is absurd. Hence $\omega_1 \cap N_3 < \omega_1 \cap N_2$. Then by (up), there exists $N_1 \in \dot{\mathcal{N}}$ s.t. $N_3 \in N_1$ and $\omega_1 \cap N_1 = \omega_1 \cap N_2$. Let $N_4 = \phi_{N_1N_2}(N_3)$. Then by (down), $N_4 \in \dot{\mathcal{N}} \cap N_2$ and $\omega_2 \cap N_4 = \phi_{N_1N_2}[\omega_2 \cap \mathrm{N}_3] = \omega_2 \cap \mathrm{N}_3$.

Corollary. Let $\dot{\mathcal{A}} = \{\omega_2 \cap N \mid N \in \dot{\mathcal{N}}\}$. Then $\dot{\mathcal{A}}$ is a simplified $(\omega_1, 1)$-morass in the sense of definition 2.6 in [V1].

Proof. We list facts related to each item.

(well founded) If $\omega_2 \cap N$ is a proper subset of $\omega_2 \cap M$, then $\omega_1 \cap N < \omega_1 \cap M$.

(homogeneous) Let $N \in \dot{\mathcal{N}}$ and $\dot{\mathcal{A}} \lceil (\omega_2 \cap N) = \{Z \in \dot{\mathcal{A}} \mid Z$ is a proper subset of $\omega_2 \cap N\}$. Then $\dot{\mathcal{A}} \lceil (\omega_2 \cap N) = \{\omega_2 \cap M \mid M \in \dot{\mathcal{N}} \cap N\}$ holds. Notice $\omega_2 \cap N$, $\omega_2 \cap M$ are of the same rank iff $\omega_1 \cap N = \omega_1 \cap M$.

(locally small) $\dot{\mathcal{N}} \cap N$ is countable.

(directed) If $N_1, N_2 \in \dot{\mathcal{N}}$, then by (cof), there exists $N \in \dot{\mathcal{N}}$ with $N_1, N_2 \in N$. This we dubbed (dir). In particular, $\omega_2 \cap N_1$, $\omega_2 \cap N_2$ are proper subsets of $\omega_2 \cap N$.

(locally almost directed): If $N \in \mathrm{zero}(\dot{\mathcal{N}})$, then $\dot{\mathcal{A}} \lceil (\omega_2 \cap N)$ is vacuously directed. If $N \in \mathrm{suc}(\dot{\mathcal{N}})$, then either $\dot{\mathcal{N}} \cap N$ satisfies (One) and so $\dot{\mathcal{A}} \lceil (\omega_2 \cap N)$ is directed, or (Two) and so has a maximal split end. If $N \in \lim(\dot{\mathcal{N}})$, then $\dot{\mathcal{A}} \lceil (\omega_2 \cap N)$ is directed.

(cover): By (cof), we have $\omega_2 = \bigcup \dot{\mathcal{A}}$.

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